The Killed Mollified Super Brownian Motion and Paracontrolled Wild Sums

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Abstract

The main goal of this work is to construct the killed rough Super Brownian Motion, a superprocess in a white noise environment first introduced by Perkowksi and Rosati. To achieve this, we use a Branching Brownian Motion and a novel intermediate process called the killed mollified Super Brownian Motion. Not only does this simplify the construction of Rosati et al., it may also be more aligned with biological intuition.
In order to show the uniqueness of the killed rough Super Brownian Motion, we construct a solution to the associated Evolution Equation, a certain Singular Stochastic Partial Differential Equation with a logistic non-linearity. This construction is carried out by introducing what we shall call Paracontrolled Wild sums, a lightweight tool which also yields the approximability of solutions and differentiability with respect to a small parameter in the initial condition.

Contents

Introduction

Superprocesses are measure-valued Markov processes that arise as scaling limits of individual based models (IBMs). They have enjoyed a lot of attention since the 1980s due to their mathematical properties and connections to biology, see [Eth00, p. xi f.] for a focused introduction and further references.
One of the most prominent examples is the so-called Super Brownian Motion (SBM), which arises e.g. as a high density limit of a Branching Random Walk (BRW) or of a Branching Brownian Motion (BBM). Naturally, a lot of different processes converge to the same Brownian limit and the resulting universality class of SBM is vast.
From a biological perspective, one should be aware of the shortcomings of each population model: For example, classical SBM does not account for heterogeneous, non-trivial environments.
One possible type of randomness of space could be white noise: White noise arises in a lot of different scenarios, for example as a scaling limit of Poisson Random Measures or of i.i.d. centred random variables with normed second moments on a dicrete grid, see [Dom09]. White noise can also be seen as the multi-dimensional generalization of the distributional derivative of Brownian Motion, hence gives rise to so-called Stochastic Partial Differential Equations (SPDEs). Problematic however is the fact that white noise suffers from rather poor (in fact negative) regularity. This leads to some nonlinear functionals appearing in SPDEs to be ill-defined. Such equations are called Singular SPDEs (SSPDEs) and can only be treated rigorously since the works of Gubinelli, Perkowski et al., [GIP15], [GP17] and Hairer [Hai14]; see also [GIP15] for a survey on various earlier approaches.
Recently, Perkowski and Rosati constructed the rough SBM (rSBM), an SBM in a spatial white noise environment, using a BRW in a random environment (BRWRE); and also a variant with killing at the boundary of the domain (killed rSBM, krSBM), [PR19], [Ros19]. They subsequently showed that the evolution of the rSBM differs from the evolution of the classical SBM, suggesting that the environment, in which the SBM lives, may cause artifacts. More precisely, they showed that the krSBM in $d = 2$ is persistent,

in contrast to the classical case [Eth00, Theorem 2.18] and the case including a space-time noise [MX07].
Inspired by the works of Rosati et al., we investigate a BBM in a random environment (BBMRE) with killing at the boundary (killed BBMRE, kBBMRE). We first consider a BBM in a mollified white noise environment, such that the environment is given by a smooth function rather than a distribution. We then introduce an intermediate process called the killed mollified SBM (kmSBM), which arises if we take the superprocess limit while leaving the mollification untouched. It turns out that the kmSBM shares a lot of its properties with the krSBM. Particularly, we show that the kmSBM also converges to the krSBM, if we let the mollification of the environment vanish. In fact, it may also be biologically more reasonable to differentiate between the parameters concerning the population and those concerning the environment; see also below.
Just as the SBM is linked to the heat equation, the killed mollified SBM is associated to the mollified Parabolic Anderson Model (mollified PAM). Understanding the behaviour and convergence of solutions to the mollified PAM to solutions of the unmollified PAM, an SSPDE, is therefore crucial for understanding the convergence of the kmSBM to the krSBM.
A different motivation for this research stems from the fact that SSPDEs often need to be renormalized by formally subtracting infinity (made rigorous by the papers mentioned above through a renormalizing sequence). By considering an associated IBM, we also investigate how the abstract renormalization affects a rather intuitive population.

The above outline first requires the analysis of the Parabolic Anderson Model using the methods of [GIP15] and extensions due to [CvZ19] to include Dirichlet boundary conditions, see Section 1. There we also introduce Paracontrolled Wild sums in order to construct a solution to the Evolution Equation for the killed rough Super Brownian Motion.
In Section 2 we construct the killed mollified SBM using essentially classical arguments which are outlined in [Eth00, Section 1.5].
In Section 3 we construct the krSBM using the kmSBM. The main difficulty lies in the fact that one needs to be careful when it comes to the domain of the associated martingale problem. Here the results of Section 1 are essential.
Finally in Section 4 we prove the persistence of the kmSBM, which differentiates it from the classical SBM but also suggests that we can use this simpler model to understand the behaviour of the krSBM.

Discussion

In this subsection we would like to discuss our reasons for constructing an intermediate process, rather than going directly from the kBBMRE to the krSBM.
In our IBM we introduce two parameters: $n, m \in \mathbb{N}$, where n concerns the population and m the environment. The function ξ_m shall denote a mollification of white noise and $c_m = O(\log(m))$ will be the renormalizing sequence. The branching mechanism of the

BBM will then be given by $\Phi_{n,m}(s) := \zeta_{n,m}^{+1} s^2 + \zeta_{n,m}^{-1}$, where

$$\zeta_{n,m}^{+1}(x) = \frac{1}{2}\left(1 + \frac{\xi_m(x) - c_m}{n}\right), \quad \zeta_{n,m}^{-1}(x) = \frac{1}{2}\left(1 - \frac{\xi_m(x) - c_m}{n}\right).$$

This means that the probability of branching into two particles upon death at the location $x \in (0, L)^2$ is $\zeta_{n,m}^{+1}(x)$ and the probability of not branching is $\zeta_{n,m}^{-1}(x)$.
By inspecting the branching probabilities, we see that indeed m concerns only the environment $\xi_m - c_m$ and n the interaction strength of the population with it. The coefficient n will also affect the mass of each individual, their birth/death rates and the size of the population. Hence from a biological perspective it is quite natural to consider n and m as separate.
To put it differently: For fixed $m \in \mathbb{N}$ and large n, we consider a population model where the environment is not perfectly homogeneous, but rather exhibits small scale fluctuations around $1/2$.
What is more, one of the main textbook examples of an empirical process following the PAM is the growth of asexually reproducing plankton, [CM94, Chapter 1]. The growth rate depends on several factors, e.g. the salinity and the temperature of the water. However, as water diffuses and conducts heat, we argue that the environment is already mollified. Consequently, it follows that the killed mollified SBM may be a reasonable model in itself as well.
From a mathematical perspective, we run into the following problems if we were to set $m = n$:
First and foremost, the probability generating function will be ill-defined. Previously, we could fix m and consider n large enough such that $\zeta_{n,m}^{+1} \in (0, 1)$. Now the appropriate bound reads $\|\xi_m\|_{L^\infty} \lesssim m^{1+\varepsilon}$ a.s. for any $\varepsilon > 0$, hence we cannot simply choose $n = m$ large. Also, the martingale problem for our kBBMRE yields expressions involving

$$\frac{1}{n} \nabla T_s^m \phi^T T_s^m \phi,$$

where T^m denotes the solution operator of

$$\begin{cases} (\partial_t - \Delta) u_m = (\xi_m - c_m) u_m & \text{in } (0, T) \times (0, L)^2, \\ u_m(0) = u_0^m \text{ in } [0, L]^2, \ u_m = 0 \text{ on } [0, T] \times \partial[0, L]^2. \end{cases}$$

The limit, though, will in general not be differentiable, hence $\nabla T^m \phi$ diverges as $m \to \infty$. Let us remark that in [PR19], Perkowski and Rosati use the fact that multiplying by n^{-1} induces a gain of regularity in the discrete case, see [PR19, Lemma D.2].
Keeping in mind that the killed mollified SBM exhibits a lot of the same properties as the killed rough SBM, we shall focus more on the kmSBM as a means of simplifying the construction. In itself this is not entirely trivial: First we work with the state space $(0, L)^2$ to leverage the connection to the theory that was already developed by [CvZ19] on the (Neumann) white noise defined on $(0, L)^2$. However, the square comes with a non-smooth boundary, which complicates the PDE part of our analysis.

What is more, we want to show that the kmSBM also converges to the krSBM, if we let $m \to \infty$. This leads us to developing the solution theory for the continuous Dirichlet Parabolic Anderson Model on the square, which needs an extension of the theory considered in [GIP15].

In fact, by being more careful, we can see a connection between the model considered in [PR19], [Ros19], and our model. Of course, Random Walks are related to Brownian Motions by Donsker's theorem. The branching mechanism in Perkowski and Rosati's model is given by a BRW in a random grid environment. For each grid point x there is a centred i.i.d. random variable $m^{-d/2}\phi_m(x)$ with normed second moment which determines the branching mechanism through the rate $|\phi_m(x) - c_m|$ and the branching probabilities

$$\theta_m^{+1}(x) := \frac{(\phi_m(x) - c_m)^+}{|\phi_m(x) - c_m|}, \quad \theta_m^{-1}(x) := \frac{(\phi_m(x) - c_m)^-}{|\phi_m(x) - c_m|}.$$

We recall that $\|\xi_m\|_{L^\infty} \lesssim m^{1+\varepsilon}$ for any $\varepsilon > 0$. So instead of setting $n = m$ in our model, let us formally set $n = |\xi_m|$. This yields

$$\zeta^{+1}_{|\xi_m|,m}(x) = \frac{1}{2}\left(\frac{2\xi_m^+ - c_m}{|\xi_m|}\right), \quad \zeta^{-1}_{|\xi_m|,m}(x) = \frac{1}{2}\left(\frac{2\xi_m^- + c_m}{|\xi_m|}\right).$$

Finally, our model only converges after rescaling time by $n = |\xi_m|$, which corresponds to the rate in Perkowski and Rosati's model. Note that c_m is of lower order compared to $\|\xi_m\|_{L^\infty}$. Hence, the branching mechanisms above are similar, at least formally and for large m.

Contributions

In the following, we list some of the novel insights we derived in this work:

- Extended the solution theory for the PAM of [GIP15] to include Dirichlet boundary conditions.
- Derived the solution theory for the 'Evolution Equation for the killed rough SBM', a non-linear SSPDE with quadratic RHS, by using Paracontrolled Wild sums.
- Identified the killed mollified SBM, which shares a number of its properties with the krSBM, but comes with a simpler construction and may be biologically more reasonable.
- Identified natural scaling parameters to show convergence of the kBBMRE to the killed mollified SBM and persistence: n: Population size, mass rescaling, time rescaling, interaction strength with environment, m: Environmental smoothness or -diversity, tilting due to renormalization.

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1 Framework of Paracontrolled Calculus

1.1 Motivation

The theory of Paracontrolled Calculus was developed by Gubinelli, Perkowski et al. [GIP15], [GP17] as a toolset for analysing SSPDEs. Take for example the Parabolic Anderson Model (PAM):

$$(\partial_t - \Delta)u = \ '\xi u',$$

where ξ is a realisation of white noise. The product $'\xi u'$ is ill-defined: In dimension $d = 2$, ξ will have the (Besov-Hölder-) regularity $\alpha < -1$ and we can consequently expect u to be at most of regularity $\alpha + 2$ by parabolic regularity. Even if we were to interpret the derivatives in a distributional sense, products of distributions cannot easily be defined if their regularities do not add up to a positive constant. It turns out that there is a paraproduct decomposition $u\xi = u \varolessthan \xi + \xi \varolessthan u + u \odot \xi$ and the term which is ill-defined will be the resonant product $\odot$.
The idea is to propose the paracontrolled Ansatz $u = u \varolessthan \vartheta + u^\sharp$ with $u^\sharp$ of regularity $2(\alpha+2)$. Then, $u^\sharp \odot \xi$ is well-defined, since we may assume $3\alpha+4 > 0$. The distribution ϑ can be chosen to be $\vartheta = (1-\Delta)^{-1}\xi$, subject to the proposed regularity of $u^\sharp$ after considering the equation. By using a commutator result between the paraproduct $\varolessthan$ and the resonant product $\odot$, we arrive at the ill-defined term $\xi \odot \vartheta$. If we now were to consider the mollified noises $(\xi_m)_{m\in\mathbb{N}}$ and $(\vartheta_m)_{m\in\mathbb{N}}$, then $\xi_m \odot \vartheta_m$ would be well-defined, but divergent as $m \to \infty$. However, by subtracting a renormalizing sequence c_m, we get $\xi_m \odot \vartheta_m - c_m \to \Xi$ for some distribution Ξ of regularity $2\alpha+2$. The tupel $\boldsymbol{\xi} = (\xi, \Xi)$ is called an enhancement of white noise.
We can then prove that solutions to the mollified problem

$$(\partial_t - \Delta)u_m = (\xi_m - c_m)u_m$$

do converge to some function u. Below we will make the above rigorous and consider a number of questions around the PAM and its paracontrolled solutions.
Let us point out that the regularity of white noise, $-d/2 - \varepsilon$, $\varepsilon > 0$, depends on the dimension d, [Ver11], while the gain of regularity due to the equation is fixed. This implies that the solution theory for SSPDEs is highly dimension dependent. For this reason, we will restrict our attention to $d = 2$ in most of Section 1, if not otherwise specified.

1.2 Paracontrolled Calculus with Boundary Conditions

In this section we review the theory of Paracontrolled Calculus with Dirichlet boundary conditions developed in [CvZ19]. For the reader's convenience, we also introduce all of the necessary notions and ideas. This is a shortened version of [CvZ19, Section 4].
We first introduce some useful notation for working with multidimensional odd or even functions. Let $d \in \mathbb{N}$, $L > 0$ and $\mathfrak{q} \in \{-1,1\}^d$. We define $(\prod \mathfrak{q}) = \prod_{i=1}^d \mathfrak{q}_i$ and for $x \in \mathbb{R}^d$

the Hadamard product $\mathfrak{q} \circ x = (\mathfrak{q}_1 x_1, \ldots, \mathfrak{q}_d x_d)$. Let $f : [-L, L]^d \to \mathbb{C}$. We say that f is odd, if $f(x) = (\prod \mathfrak{q}) f(\mathfrak{q} \circ x)$ for any $\mathfrak{q} \in \{-1, 1\}^d$ and $x \in [-L, L]^d$ and that it is even, if $f(x) = f(\mathfrak{q} \circ x)$. If f is periodic and odd, then $f(x) = 0$, whenever $x \in \partial[0, L]^2$.
Let now $f : [0, L]^d \to \mathbb{C}$. We define the odd extension of f by $\tilde{f} : [-L, L]^d \to \mathbb{C}$, $\tilde{f}(\mathfrak{q} \circ x) = (\prod \mathfrak{q}) f(x)$ and the even extension by $\bar{f} : [-L, L]^d \to \mathbb{C}$, $\bar{f}(\mathfrak{q} \circ x) = f(x)$ for any $x \in [0, L]^d$ and $\mathfrak{q} \in \{-1, 1\}^d$. Let for $k \in \mathbb{N}_0^d$, $\nu_k = 2^{-1/2 \mathrm{Card}\{i | k_i = 0\}}$. Let $\mathfrak{d}_k : [0, L]^d \to \mathbb{R}$, $\mathfrak{n}_k : [0, L]^d \to \mathbb{R}$ and $\mathfrak{e}_k : [-L, L]^d \to \mathbb{C}$ be given by

$$\mathfrak{d}_k(x) = \left(\frac{2}{L}\right)^{\frac{d}{2}} \prod_{i=1}^{d} \sin\left(\frac{\pi}{L} k_i x_i\right), \quad \mathfrak{n}_k(x) = \nu_k \left(\frac{2}{L}\right)^{\frac{d}{2}} \prod_{i=1}^{d} \cos\left(\frac{\pi}{L} k_i x_i\right),$$
$$\mathfrak{e}_k(x) = \left(\frac{1}{2L}\right)^{\frac{d}{2}} \exp\left(\frac{\pi \mathfrak{i}}{L} \langle k, x \rangle\right).$$

Note that the application of $\tilde{\cdot}$ to $\mathfrak{d}_k$ and the application of $\bar{\cdot}$ to $\mathfrak{n}_k$ merely changes the domain of the functions. Then $\mathfrak{n}_k$ is even and $\mathfrak{d}_k$ is odd on $[-L, L]^d$, both are periodic and smooth. We define the torus $\mathbb{T}_{2L}^d = (\mathbb{R}/(2L\mathbb{Z}))^d$. If a function on $[-L, L]^d$ can be extended periodically to $\mathbb{R}^d$, then by a slight abuse of notation we will also consider it to be a function on $\mathbb{T}_{2L}^d$. Let $f : \mathbb{T}_{2L}^d \to \mathbb{C}$. We define the Fourier transform with the scaling

$$\mathcal{F}_{\mathbb{T}_{2L}^d} f(k) = \langle f, \mathfrak{e}_k \rangle_{L^2([-L,L]^d, \mathbb{C})} = \left(\frac{1}{2L}\right)^{\frac{d}{2}} \int_{\mathbb{T}_{2L}^d} f(x) \exp\left(-\frac{\pi \mathfrak{i}}{L} \langle k, x \rangle\right) dx,$$

where $k \in \mathbb{Z}^d$.

Definition 1.1 [CvZ19, Definition 4.4]
We define for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$,

$$S(\mathbb{T}_{2L}^d, \mathbb{F}) = C^\infty(\mathbb{T}_{2L}^d, \mathbb{F}), \quad S_{\mathfrak{d}}([0, L]^d, \mathbb{F}) = \{\phi \in C^\infty([0, L]^d, \mathbb{F}) | \tilde{\phi} \in S(\mathbb{T}_{2L}^d, \mathbb{F})\},$$
$$S_{\mathfrak{n}}([0, L]^d, \mathbb{F}) = \{\phi \in C^\infty([0, L]^d, \mathbb{F}) | \bar{\phi} \in S(\mathbb{T}_{2L}^d, \mathbb{F})\},$$

equipped with the Schwartz seminorms. The continuous dual spaces are denoted by $S'(\mathbb{T}_{2L}^d, \mathbb{F})$, $S'_{\mathfrak{d}}([0, L]^d, \mathbb{F})$ *and* $S'_{\mathfrak{n}}([0, L]^d, \mathbb{F})$, *and are equipped with the weak*-topologies.*

Theorem 1.2 [CvZ19, Theorem 4.5]
The following hold:

(i) Every $w \in S(\mathbb{T}_{2L}^d, \mathbb{C})$, $\phi \in S_{\mathfrak{d}}([0, L]^d, \mathbb{C})$ *and* $\psi \in S_{\mathfrak{n}}([0, L]^d, \mathbb{C})$ *can be represented as*

$$w = \sum_{k \in \mathbb{Z}^d} a_k \mathfrak{e}_k, \quad \phi = \sum_{k \in \mathbb{N}^d} b_k \mathfrak{d}_k, \quad \psi = \sum_{k \in \mathbb{N}_0^d} c_k \mathfrak{n}_k, \tag{1}$$

with $(a_k)_{k \in \mathbb{Z}^d}$, $(b_k)_{k \in \mathbb{N}^d}$ *and* $(c_k)_{k \in \mathbb{N}_0^d}$ *complex-valued sequences. Further for any* $n \in \mathbb{N}$,

$$\sup_{k \in \mathbb{Z}^d} (1 + |k|)^n |a_k| < \infty, \quad \sup_{k \in \mathbb{N}^d} (1 + |k|)^n |b_k| < \infty, \quad \sup_{k \in \mathbb{N}_0^d} (1 + |k|)^n |c_k| < \infty. \tag{2}$$

What is more, $a_k = \langle w, \mathfrak{e}_k \rangle_{L^2([-L,L]^d,\mathbb{C})}$, $b_k = \langle \phi, \mathfrak{d}_k \rangle_{L^2([0,L]^d,\mathbb{C})}$ *and as well* $c_k = \langle \psi, \mathfrak{n}_k \rangle_{L^2([0,L]^d,\mathbb{C})}$. *Conversely, if there are sequences* $(a_k)_{k\in\mathbb{Z}^d}$, $(b_k)_{k\in\mathbb{N}^d}$ *and* $(c_k)_{k\in\mathbb{N}_0^d}$ *that satisfy* (2) *respectively, then the expressions in* (1) *converge in the respective spaces.*

(ii) Every $w \in S'(\mathbb{T}^d_{2L}, \mathbb{C}), \phi \in S'_{\mathfrak{d}}([0,L]^d, \mathbb{C})$ *and* $\psi \in S'_{\mathfrak{n}}([0,L]^d, \mathbb{C})$ *can be represented as*

$$w = \sum_{k\in\mathbb{Z}^d} a_k \mathfrak{e}_k, \quad \phi = \sum_{k\in\mathbb{N}^d} b_k \mathfrak{d}_k, \quad \psi = \sum_{k\in\mathbb{N}_0^d} c_k \mathfrak{n}_k, \tag{3}$$

with $(a_k)_{k\in\mathbb{Z}^d}$, $(b_k)_{k\in\mathbb{N}^d}$ *and* $(c_k)_{k\in\mathbb{N}_0^d}$ *complex-valued sequences. Further for* some $n \in \mathbb{N}$,

$$\sup_{k\in\mathbb{Z}^d} (1+|k|)^{-n}|a_k| < \infty, \quad \sup_{k\in\mathbb{N}^d} (1+|k|)^{-n}|b_k| < \infty, \quad \sup_{k\in\mathbb{N}_0^d} (1+|k|)^{-n}|c_k| < \infty. \tag{4}$$

What is more, $a_k = \langle w, \mathfrak{e}_k \rangle$, $b_k = \langle \phi, \mathfrak{d}_k \rangle$ *and as well* $c_k = \langle \psi, \mathfrak{n}_k \rangle$. *Conversely, if there are sequences* $(a_k)_{k\in\mathbb{Z}^d}, (b_k)_{k\in\mathbb{N}^d}$ *and* $(c_k)_{k\in\mathbb{N}_0^d}$ *that satisfy* (4) *respectively, then the expressions in* (3) *converge in the respective spaces.*

Remark 1.3
We see that $(\mathfrak{d}_k)_{k\in\mathbb{N}^d}$ and $(\mathfrak{n}_k)_{k\in\mathbb{N}_0^d}$ take the roles of the Fourier base $(\mathfrak{e}_k)_{k\in\mathbb{Z}^d}$ for functions with Dirichlet- or Neumann boundary conditions, or respectively for odd or even functions.

Note that $w \in S(\mathbb{T}^d_{2L}, \mathbb{C})$ is odd if and only if $\langle w, \mathfrak{e}_{\mathfrak{q}\circ k} \rangle = (\prod \mathfrak{q}) \langle w, \mathfrak{e}_k \rangle$ for any $k \in \mathbb{Z}^d$ and $\mathfrak{q} \in \{-1,1\}^d$. This motivates the following:

Definition 1.4 [CvZ19, Definition 4.6]
For $u \in S'_{\mathfrak{d}}([0,L]^d)$ *we define* $\tilde{u} \in S'(\mathbb{T}^d_{2L})$ *given by* $\tilde{u} = \sum_{k\in\mathbb{N}} \langle u, \mathfrak{d}_k \rangle \tilde{\mathfrak{d}}_k$. *For* $v \in S'_{\mathfrak{n}}([0,L]^d)$ *we define* $\bar{v} \in S'(\mathbb{T}^d_{2L})$ *given by* $\bar{v} = \sum_{k\in\mathbb{N}} \langle v, \mathfrak{n}_k \rangle \bar{\mathfrak{n}}_k$. *A distribution* $w \in S'(\mathbb{T}^d_{2L})$ *is called odd, if* $\langle w, \mathfrak{e}_{\mathfrak{q}\circ k} \rangle = (\prod \mathfrak{q}) \langle w, \mathfrak{e}_k \rangle$ *for any* $k \in \mathbb{Z}^d$ *and* $\mathfrak{q} \in \{-1,1\}^d$. *Similarly,* w *is called even, if* $\langle w, \mathfrak{e}_{\mathfrak{q}\circ k} \rangle = \langle w, \mathfrak{e}_k \rangle$ *for any* $k \in \mathbb{Z}^d$ *and* $\mathfrak{q} \in \{-1,1\}^d$.

We have by partial integration, $\mathcal{F}_{\mathbb{T}^d_{2L}}(\partial^\alpha w)(k) = (k\pi\mathfrak{i}/L)^\alpha \mathcal{F}_{\mathbb{T}^d_{2L}}(w)(k)$ for any $\alpha \in \mathbb{N}_0^d$. This motivates the following:

Definition 1.5 [CvZ19, Definition 4.8]
Let $\varepsilon > 0$, $\tau : \mathbb{R}^d \to \mathbb{R}$, $\sigma : [0,\infty)^d \to \mathbb{R}$, $w \in S'(\mathbb{T}^d_{2L}, \mathbb{C})$, $u \in S'_{\mathfrak{d}}([0,L]^d, \mathbb{C})$ *and*

$v \in S'_{\mathfrak{n}}([0,L]^d, \mathbb{C})$. *We define formally the Fourier multipliers*

$$\tau(\varepsilon D)w = \sum_{k\in\mathbb{Z}^d} \tau(\varepsilon k/L)\, \langle w, \mathfrak{e}_k\rangle\, \mathfrak{e}_k, \quad \sigma(\varepsilon D)u = \sum_{k\in\mathbb{N}^d} \sigma(\varepsilon k/L)\, \langle u, \mathfrak{d}_k\rangle\, \mathfrak{d}_k,$$
$$\sigma(\varepsilon D)v = \sum_{k\in\mathbb{N}_0^d} \sigma(\varepsilon k/L)\, \langle v, \mathfrak{n}_k\rangle\, \mathfrak{n}_k.$$

The terminology 'formally' refers to the requirement to verify the respective condition in (4) *for each case.*

It holds by [CvZ19, (30)], that for any $u \in S'_{\mathfrak{d}}([0,L]^d, \mathbb{C})$ and $v \in S'_{\mathfrak{n}}([0,L]^d, \mathbb{C})$,

$$\widetilde{\sigma(D)u} = \bar{\sigma}(D)\tilde{u}, \quad \overline{\sigma(D)v} = \bar{\sigma}(D)\bar{v}. \tag{5}$$

Let (χ, ρ) be a dyadic partition of unity, i.e. let χ, ρ be non-negative, radially symmetric, smooth functions on $\mathbb{R}^d$ such that: χ is supported in a ball and ρ is supported in an annulus. Further, if we set $\rho_{-1} := \chi$ and $\rho_j := \rho(2^{-j}\cdot)$ for $j \geq 0$, then for any $y \in \mathbb{R}^d$, $j, k \geq -1$, we suppose

$$\sum_{j\geq -1} \rho_j(y) = 1, \quad \frac{1}{2} \leq \sum_{j\geq -1} \rho_j(y)^2 \leq 1, \quad |j-k| \geq 2 \Rightarrow \operatorname{supp}(\rho_j) \cap \operatorname{supp}(\rho_k) = \emptyset.$$

Such functions exist by [BCD11, Proposition 2.10]. For $j \geq -1$, we define the Littlewood-Paley blocks given by the Fourier multipliers $\Delta_j := \rho_j(D)$ acting respectively on functions of the different spaces considered in Definition 1.5.
To give some context here, let us add that the Littlewood-Paley blocks act precisely as a decomposition into functions with localized support in Fourier space. It is well known that certain decays in Fourier space correspond to regularity. In particular, we have already seen above that regularity is crucial for our analysis. Therefore it is natural to use Littlewood-Paley blocks as a means of consistently quantifying the regularity of objects. This is the idea of the following definition of Besov spaces. Since we need to re-develop central ideas of classical Besov space theory in the presence of boundary conditions, we introduce all notions simultaneously.

Definition 1.6 [CvZ19, Definition 4.9]
Let $a_{d,p} := 2^{-d/p}$ *if* $p < \infty$ *and* $a_{d,\infty} := 1$. *Let* $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$. *We define the periodic Besov spaces by*

$$B^\alpha_{p,q}(\mathbb{T}^d_{2L}) = \{u \in S'(\mathbb{T}^d_{2L}, \mathbb{R}) | \|u\|_{B^\alpha_{p,q}(\mathbb{T}^d_{2L})} := \|(2^{j\alpha}\|\Delta_j u\|_{L^p([-L,L]^d,\mathbb{C})})_{j\geq -1}\|_{l^q} < \infty\}.$$

Further, we define the Dirichlet Besov spaces by

$$B^{\mathfrak{d},\alpha}_{p,q}([0,L]^d) = \{u \in S'_{\mathfrak{d}}([0,L]^d, \mathbb{R}) | \|u\|_{B^{\mathfrak{d},\alpha}_{p,q}([0,L]^d)} := a_{d,p}\|\tilde{u}\|_{B^\alpha_{p,q}(\mathbb{T}^d_{2L})} < \infty\}.$$

Finally, we define the Neumann Besov spaces by

$$B^{\mathfrak{n},\alpha}_{p,q}([0,L]^d) = \{u \in S'_{\mathfrak{n}}([0,L]^d, \mathbb{R}) | \|u\|_{B^{\mathfrak{n},\alpha}_{p,q}([0,L]^d)} := a_{d,p}\|\bar{u}\|_{B^\alpha_{p,q}(\mathbb{T}^d_{2L})} < \infty\}.$$

Each of them is equipped with the norm appearing in the corresponding definition. We may write $B^{\alpha}_{p,q}$, $B^{\mathfrak{d},\alpha}_{p,q}$, $B^{\mathfrak{n},\alpha}_{p,q}$ and the other cases analogously if there is no ambiguity.

Theorem 1.7 [CvZ19, below Definition 4.9]
The Besov-, Dirichlet Besov- and Neumann Besov- spaces are all Banach spaces with corresponding norms. Also, the resulting spaces are independent of the chosen dyadic partition of unity.

Proof
The assertion for the Besov space on $\mathbb{R}^d$ is given in [BCD11, Theorem 2.72] and carries over to the periodic setting due to the extendability of distributions on the torus to periodic distributions on the full space. This subsequently carries over to the other spaces by the definition via odd or even extensions. For the independence of (χ, ρ), see [BCD11, Corollarly 2.70].

□

Definition 1.8 [CvZ19, below Definition 4.9]
Let $\alpha \in \mathbb{R}$. We write $\mathcal{C}^{\alpha} := B^{\alpha}_{\infty,\infty}(\mathbb{T}^d_{2L})$, $\mathcal{C}^{\alpha}_{\mathfrak{n}} := B^{\mathfrak{n},\alpha}_{\infty,\infty}([0,L]^d)$, $\mathcal{C}^{\alpha}_{\mathfrak{d}} := B^{\mathfrak{d},\alpha}_{\infty,\infty}([0,L]^d)$ and $H^{\alpha}_0 := B^{\mathfrak{d},\alpha}_{2,2}([0,L]^d)$.

To see the intuition behind the notation of Definition 1.8, we need the following:

Theorem 1.9 [CvZ19, Theorem 4.15]
Let $\alpha \in \mathbb{R}$. We have for any function f,

$$\|f\|_{H^{\alpha}_0} \simeq \sqrt{\sum_{k\in\mathbb{N}^d_0} \left(1 + |k/L|^2\right)^{\alpha} \langle f, \mathfrak{d}_k\rangle^2}.$$

The above implies that $H^0_0 = L^2([0,L]^d)$, since $(\mathfrak{d}_k)_{k\in\mathbb{N}^d_0}$ forms an orthonormal basis in $L^2([0,L]^d, \mathbb{R})$, [CvZ19, Lemma 4.3]. In particular, H^{α}_0, $\alpha > 0$, coincides with the classical Sobolev spaces with Dirichlet boundary conditions, see [CvZ19, Theorem 4.16].

Theorem 1.10 [CvZ19, Theorem 4.7 (c)]
We have

$$\begin{aligned}\tilde{S}'_{\mathfrak{d}}(\mathbb{T}^d_{2L}, \mathbb{R}) &:= \{\tilde{u} | u \in S'_{\mathfrak{d}}([0,L]^d, \mathbb{R})\} = \{w \in S'(\mathbb{T}^d_{2L}, \mathbb{R}) | w \text{ is odd}\},\\ \bar{S}'_{\mathfrak{n}}(\mathbb{T}^d_{2L}, \mathbb{R}) &:= \{\bar{u} | u \in S'_{\mathfrak{n}}([0,L]^d, \mathbb{R})\} = \{w \in S'(\mathbb{T}^d_{2L}, \mathbb{R}) | w \text{ is even}\}.\end{aligned}$$

Both spaces are closed in $S'(\mathbb{T}^d_{2L}, \mathbb{R})$. The spaces $S'(\mathbb{T}^d_{2L}, \mathbb{R})$, $S'_{\mathfrak{d}}([0,L]^d)$, $S'_{\mathfrak{n}}([0,L]^d)$ are weak-sequentially complete.*

In the absence of boundary conditions, $B^{\alpha}_{\infty,\infty}$ coincides with the Hölder spaces C^{α} if $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}_0$. By combining this with Theorem 1.10, we see that Definition 1.8 captures that

$\mathcal{C}_{\mathfrak{d}}^{\alpha}$ and $\mathcal{C}_{\mathfrak{n}}^{\alpha}$ are the spaces of Hölder-continuous functions with Dirichlet- or respectively Neumann boundary conditions.
The Hölder-Besov norm can also be used to control the L^∞-norm of functions:

Lemma 1.11
It holds that for any $\varepsilon > 0$, $\|u\|_{L^\infty} \lesssim \|u\|_{\mathcal{C}_{\mathfrak{d}}^{\varepsilon}}$ if $u \in \mathcal{C}_{\mathfrak{d}}^{\varepsilon}$ and $\|u\|_{L^\infty} \lesssim \|u\|_{\mathcal{C}_{\mathfrak{n}}^{\varepsilon}}$ if $u \in \mathcal{C}_{\mathfrak{n}}^{\varepsilon}$.

Proof
As stated in [GIP15, Appendix A] it holds that $\|\tilde{u}\|_{L^\infty(\mathbb{T}_{2L}^d)} \lesssim \|\tilde{u}\|_{\mathcal{C}^\varepsilon}$ for any $\varepsilon > 0$ and $u \in \mathcal{C}_{\mathfrak{d}}^{\varepsilon}$. Consequently, $\|u\|_{L^\infty([0,L]^2)} = \|\tilde{u}\|_{L^\infty(\mathbb{T}_{2L}^d)} \lesssim a_{d,\infty}^{-1}\|u\|_{\mathcal{C}_{\mathfrak{d}}^{\varepsilon}}$. The other case is analogous.

□

Theorem 1.12 [CvZ19, Theorem 4.17]
Let $p, q \in [1, \infty]$ and $\alpha < \beta \in \mathbb{R}$. Then, $B_{p,q}^{\beta}$ is compactly embedded in $B_{p,q}^{\alpha}$, $B_{p,q}^{\mathfrak{n},\beta}$ is compactly embedded in $B_{p,q}^{\mathfrak{n},\alpha}$ and $B_{p,q}^{\mathfrak{d},\beta}$ is compactly embedded in $B_{p,q}^{\mathfrak{d},\alpha}$.

Theorem 1.13 [CvZ19, Theorem 4.19]
Let $\gamma \in \mathbb{R}$ and $m \geq 0$. Let $\sigma : \mathbb{R}^d \to \mathbb{R}$ be such that $|\sigma(x)| \lesssim (1 + |x|)^{-m}$ for any $x \in \mathbb{R}^d$. Then

$$\|\sigma(D)w\|_{H_0^{\gamma+m}} \lesssim \|w\|_{H_0^{\gamma}}.$$

Let $\gamma, m \in \mathbb{R}$. Let $\sigma : \mathbb{R}^d \to \mathbb{R}$ be such that $\sigma \in C^\infty(\mathbb{R}^d \setminus \{0\})$. Assume that for any $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq 2\lfloor 1 + d/2 \rfloor$ there exists some $C_\alpha > 0$ such that $|\partial^\alpha \sigma(x)| \leq C_\alpha |x|^{-m-|\alpha|}$ for any $x \neq 0$. Then

$$\|\sigma(D)w\|_{\mathcal{C}_{\mathfrak{n}}^{\gamma+m}([0,L]^d)} \lesssim \|w\|_{\mathcal{C}_{\mathfrak{n}}^{\gamma}([0,L]^d)}.$$

Remark 1.14
Let $\sigma(x) := (1 + \pi^2|x|^2)^{-1}$. Then formally $\sigma(D) = (1 - \Delta)^{-1}$ and indeed σ satisfies all of the assumptions of Theorem 1.13 with $m = 2$, see [CvZ19, 4.11 and 4.20].

A fundamental result by Schwartz states that there is no canonical way of defining a product of distributions. However, Bony realised that it is possible to define such products if the regularities add up to a positive constant. To make this precise we need to introduce paraproducts and resonant products between distributions.

Definition 1.15 [CvZ19, Definition 4.23]
Let $u \in S'_{\mathfrak{d}}([0,L]^d) \cup S'_{\mathfrak{n}}([0,L]^d)$ *and* $v \in S'_{\mathfrak{n}}([0,L]^d)$. *We write formally*

$$u \otimes v = \sum_{\substack{i,j \geq -1 \\ i \leq j-2}} \Delta_i u \Delta_j v, \quad u \odot v = \sum_{\substack{i,j \geq -1 \\ |i-j| \leq 1}} \Delta_i u \Delta_j v$$

and call $\otimes$ *the paraproduct and* $\odot$ *the resonant product. This yields the decomposition* $uv := u \otimes v + v \otimes u + u \odot v$. *The terminology 'formally' refers to the requirement to verify the assumptions of Theorem 1.16 below.*

It turns out that paraproducts are always defined irrespectively of the regularities involved, while the resonant product needs regularity assumptions. Before we proceed, we need to make one more important observation: not all boundary conditions are compatible. Recall that the Besov spaces with boundary conditions were defined in terms of odd and even extensions. Let u, v be some functions and uv be their usual product. Then it is clear that $\overline{uv} = \bar{u}\bar{v} = \tilde{u}\tilde{v}$ and $\widetilde{uv} = \tilde{u}\bar{v} = \bar{u}\tilde{v}$. Keeping this in mind, we can formulate the Bony estimates.

Theorem 1.16 (Bony's estimates I)
Let $\alpha, \beta \in \mathbb{R}$. *Then it holds that:*

(i) If $\alpha < 0$, *then* $\|\xi \otimes u\|_{\mathcal{C}^{\alpha+\beta}_{\mathfrak{d}}} \lesssim_{\alpha,\beta} \|\xi\|_{\mathcal{C}^{\alpha}_{\mathfrak{n}}} \|u\|_{\mathcal{C}^{\beta}_{\mathfrak{d}}}$.

(ii) It holds that for $u \in S'_{\mathfrak{d}}([0,L]^2)$, $\|u \otimes \xi\|_{\mathcal{C}^{\alpha}_{\mathfrak{d}}} \lesssim_{\alpha} \|u\|_{L^\infty} \|\xi\|_{\mathcal{C}^{\alpha}_{\mathfrak{n}}}$.

(iii) If $\alpha + \beta > 0$, *then* $\|u \odot \xi\|_{\mathcal{C}^{\alpha+\beta}_{\mathfrak{d}}} \lesssim_{\alpha,\beta} \|\xi\|_{\mathcal{C}^{\alpha}_{\mathfrak{n}}} \|u\|_{\mathcal{C}^{\beta}_{\mathfrak{d}}}$.

Proof
It holds that $\widetilde{\xi \otimes u} = \bar{\xi} \otimes \tilde{u}$, $\widetilde{u \otimes \xi} = \tilde{u} \otimes \bar{\xi}$ and $\widetilde{u \odot \xi} = \tilde{u} \odot \bar{\xi}$. The claims now follow from the definitions of the Dirichlet- and Neumann Besov spaces together with [GIP15, Lemma 2.1].

□

Theorem 1.17
Let $\alpha \in (0,1)$, $\beta, \gamma \in \mathbb{R}$ *such that* $\beta + \gamma < 0$ *and* $\alpha + \beta + \gamma > 0$. *Then the trilinear operator* $\mathscr{R}(f,g,h) := (f \otimes g) \odot h - f(g \odot h)$, $f \in S_{\mathfrak{d}}([0,L]^2, \mathbb{R})$, $g, h \in S_{\mathfrak{n}}([0,L]^2, \mathbb{R})$ *can be continuously extended to* $\mathcal{C}^{\alpha}_{\mathfrak{d}} \times \mathcal{C}^{\beta}_{\mathfrak{n}} \times \mathcal{C}^{\gamma}_{\mathfrak{n}}$ *such that* $\|\mathscr{R}(f,g,h)\|_{\mathcal{C}^{\alpha+\beta+\gamma}_{\mathfrak{d}}} \lesssim \|f\|_{\mathcal{C}^{\alpha}_{\mathfrak{d}}} \|g\|_{\mathcal{C}^{\beta}_{\mathfrak{n}}} \|h\|_{\mathcal{C}^{\gamma}_{\mathfrak{n}}}$.

Proof
The version on the torus with commutator denoted by C can be found in [GIP15, Lemma 2.4]. We define for $f \in S_{\mathfrak{d}}([0,L]^2, \mathbb{R})$ and $g, h \in S_{\mathfrak{n}}([0,L]^2, \mathbb{R})$, $\widetilde{\mathscr{R}(f,g,h)} := C(\tilde{f}, \bar{g}, \bar{h})$. The claimed properties then follow for f, g, h smooth by [GIP15, Lemma 2.4].

To extend the above to $\mathcal{C}^\alpha_{\mathfrak{d}} \times \mathcal{C}^\beta_{\mathfrak{n}} \times \mathcal{C}^\gamma_{\mathfrak{n}}$, we need to make sure that $\mathscr{R}$ is still well-defined beyond the smooth case.
We claim that for $f \in \mathcal{C}^\delta_{\mathfrak{d}}$, $g \in \mathcal{C}^\delta_{\mathfrak{n}}$, $\delta \in \mathbb{R}$, there exist $(f_k)_{k\in\mathbb{N}}$, $f_k \in S_{\mathfrak{d}}([0,L]^2,\mathbb{R})$, and $(g_k)_{k\in\mathbb{N}}$, $g_k \in S_{\mathfrak{n}}([0,L]^2,\mathbb{R})$ such that $f_k \to f$ in $\mathcal{C}^{\delta'}_{\mathfrak{d}}$ and $g_k \to g$ in $\mathcal{C}^{\delta'}_{\mathfrak{n}}$ for any $\delta' < \delta$.
Proof of the claim: Let $(P^{\mathrm{Dir}}_t)_{t\geq 0}$, $(P^{\mathrm{Neu}}_t)_{t\geq 0}$ denote the semigroups generated by the Dirichlet-, respectively Neumann Laplacians on $[0,L]^2$. We will see in Lemma 1.36, that for any $t \in (0,1]$, $\|P^{\mathrm{Dir}}_t f\|_{\mathcal{C}^\delta_{\mathfrak{d}}} \lesssim \|f\|_{\mathcal{C}^\delta_{\mathfrak{d}}}$ and $\|P^{\mathrm{Neu}}_t g\|_{\mathcal{C}^\delta_{\mathfrak{n}}} \lesssim \|g\|_{\mathcal{C}^\delta_{\mathfrak{n}}}$. We define $f_k := P^{\mathrm{Dir}}_{1/k} f$ and $g_k := P^{\mathrm{Neu}}_{1/k} g$, which are smooth by Lemma 1.36. By Theorem 1.12, we can extract subsequences that converge in $\mathcal{C}^{\delta'}_{\mathfrak{d}}$, and respectively in $\mathcal{C}^{\delta'}_{\mathfrak{n}}$. Denote $\lim_{k\to\infty} f_k = f' \in \mathcal{C}^{\delta'}_{\mathfrak{d}}$ and let $\varphi \in S_{\mathfrak{d}}([0,L]^2)$. It follows that

$$\langle P^{\mathrm{Dir}}_{1/k} f, \varphi\rangle = \sum_{k\in\mathbb{N}^d} \exp(-(\pi/L)^2 t|k|^2) \langle f, \mathfrak{d}_k\rangle \langle \mathfrak{d}_k, \varphi\rangle \to \sum_{k\in\mathbb{N}^d} \langle f, \mathfrak{d}_k\rangle \langle \mathfrak{d}_k, \varphi\rangle = \langle f, \varphi\rangle .$$

By using that the embedding $\mathcal{C}^{\delta'}_{\mathfrak{d}} \to S'_{\mathfrak{d}}$ is continuous, we get that $P^{\mathrm{Dir}}_{1/k} f \to f'$ in $S'_{\mathfrak{d}}$. Consequently, $\langle f', \varphi\rangle = \langle f, \varphi\rangle$ and $f = f'$ in $S'_{\mathfrak{d}}$. The same holds true for $(g_k)_{k\in\mathbb{N}}$ as well. This yields the claim.
We get for $(f_k)_{k\in\mathbb{N}}$, $(g_k)_{k\in\mathbb{N}}$, $(h_k)_{k\in\mathbb{N}}$ as above, $C(\tilde{f}_k, \bar{g_k}, \bar{h_k}) \to C(\tilde{f}, \bar{g}, \bar{h})$ in $\mathcal{C}^{\alpha'+\beta'+\gamma'}$ with $\alpha' < \alpha$, $\beta' < \beta$, $\gamma' < \gamma$ such that $\beta' + \gamma' < 0$ and $\alpha' + \beta' + \gamma' > 0$. The LHS is odd by the smoothness of the approximations and consequently, the limit is odd as well. Hence, we can extend $\mathscr{R}(f,g,h) := \widetilde{C(\tilde{f}, \bar{g}, \bar{h})}$ to $f \in \mathcal{C}^\alpha_{\mathfrak{d}}$, $g \in \mathcal{C}^\beta_{\mathfrak{n}}$, $h \in \mathcal{C}^\gamma_{\mathfrak{n}}$ and the result follows from [GIP15, Lemma 2.4].

□

If one combines Sobolev and Hölder regularities in the above, this will occasionally incur a small loss of regularity. This fact will only play a minor role.

Theorem 1.18 (Bony's estimates II)[CvZ19, Theorem 4.25][AC15, Proposition 3.1]
Let $\alpha, \beta, \delta \in \mathbb{R}$. Then it holds that

(i) *If $\alpha < 0$, then $\|\xi \circledcirc f\|_{H^{\alpha+\beta}_0} \lesssim_{\alpha,\beta} \|\xi\|_{\mathcal{C}^\alpha_{\mathfrak{n}}} \|f\|_{H^\beta_0}$.*

(ii) *If $\beta \geq 0$ and $\delta > 0$, then $\|f \circledcirc \xi\|_{H^{\alpha-\delta}_0} \lesssim_{\alpha,\beta,\delta} \|f\|_{H^\beta_0} \|\xi\|_{\mathcal{C}^\alpha_{\mathfrak{n}}}$.*

(iii) *If $\beta < 0$, then $\|f \circledcirc \xi\|_{H^{\beta+\alpha}_0} \lesssim \|f\|_{H^\beta_0} \|\xi\|_{\mathcal{C}^\alpha_{\mathfrak{n}}}$.*

(iv) *If $\alpha + \beta > 0$, then $\|f \odot \xi\|_{H^{\alpha+\beta}_0} \lesssim_{\alpha,\beta} \|f\|_{H^\beta_0} \|\xi\|_{\mathcal{C}^\alpha_{\mathfrak{n}}}$.*

(v) *If $\alpha < 0$, $\alpha + \beta > 0$, $\delta > 0$, then $\|f\xi\|_{H^{\alpha-\delta}_0} \lesssim_{\alpha,\beta,\delta} \|f\|_{H^\beta_0} \|\xi\|_{\mathcal{C}^\alpha_{\mathfrak{n}}}$.*

Theorem 1.19 [AC15, Proposition 4.3]
Let $\alpha \in (0,1)$, $\beta, \gamma \in \mathbb{R}$ such that $\beta+\gamma < 0$, $\alpha+\beta+\gamma > 0$ and let $\delta > 0$. Then the trilinear operator $\mathscr{R}(f,g,h) := (f \circledcirc g) \odot h - f(g \odot h)$, $f \in S_{\mathfrak{d}}([0,L]^2,\mathbb{R})$, $g,h \in S_{\mathfrak{n}}([0,L]^2,\mathbb{R})$ can be continuously extended to $H^\alpha_0 \times \mathcal{C}^\beta_{\mathfrak{n}} \times \mathcal{C}^\gamma_{\mathfrak{n}}$ such that $\|\mathscr{R}(f,g,h)\|_{H^{\alpha+\beta+\gamma-\delta}_0} \lesssim \|f\|_{H^\alpha_0} \|g\|_{\mathcal{C}^\beta_{\mathfrak{n}}} \|h\|_{\mathcal{C}^\gamma_{\mathfrak{n}}}$.

1.3 Regularity and Convergence of Enhanced Neumann White Noise

In this section we introduce results concerning the regularity and convergence of enhanced white noise with Neumann boundary conditions (Neumann white noise). The following can be found in [CvZ19, Section 6].

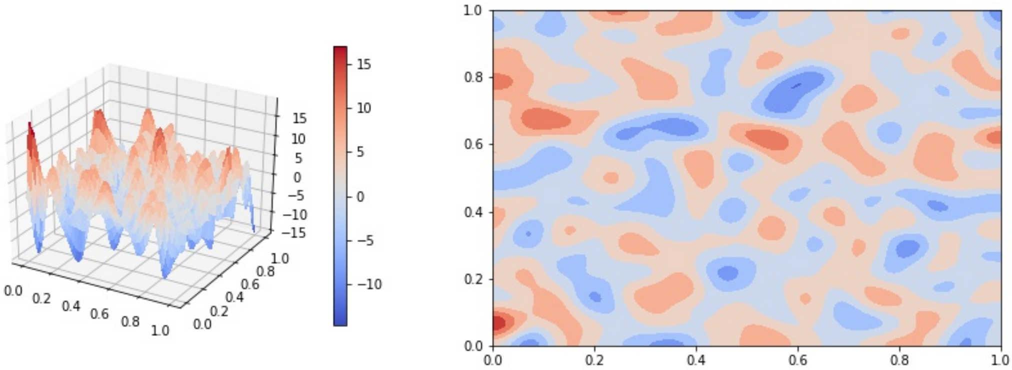

Figure 1: One realisation of mollified white noise with $L = 1$ and $m = 25$.

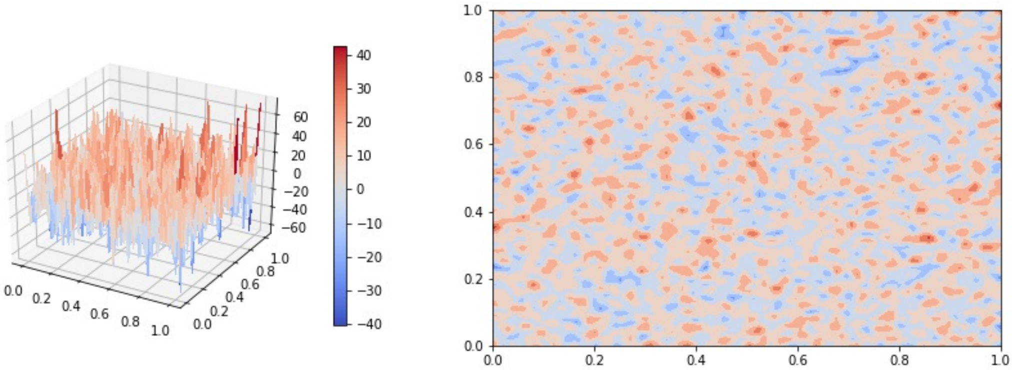

Figure 2: One realisation of mollified white noise with $L = 1$ and $m = 100$.

Definition 1.20 [CvZ19, Definition 6.1]
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A white noise on $\mathbb{R}^2$ is a random distribution $W : \Omega \to S'(\mathbb{R}^2, \mathbb{C})$ such that for any $f \in S(\mathbb{R}^2, \mathbb{C})$, $W(f)$ is a complex centred Gaussian random variable with $\overline{W(f)}^{\mathbb{C}} = W(\overline{f}^{\mathbb{C}})$ and $\mathbb{E}(W(f)\overline{W(g)}^{\mathbb{C}}) = \langle f, g\rangle_{L^2(\mathbb{R}^2,\mathbb{C})}$ for any $f, g \in S(\mathbb{R}^2, \mathbb{C})$.

We can extend W to a bounded linear operator $\mathscr{W} : L^2(\mathbb{R}^2, \mathbb{C}) \to L^2(\Omega, \mathbb{C})$ such that for any $f \in L^2(\mathbb{R}^2, \mathbb{C})$, $\mathscr{W}(f)$ is a complex centred Gaussian random variable with $\overline{\mathscr{W}(f)}^{\mathbb{C}} = \mathscr{W}(\overline{f}^{\mathbb{C}})$ and $\mathbb{E}(\mathscr{W}(f)\overline{\mathscr{W}(g)}^{\mathbb{C}}) = \langle f, g\rangle_{L^2(\mathbb{R}^2,\mathbb{C})}$. We now define the mollified Neumann white noise. Let $\tau \in C_c^\infty(\mathbb{R}^2, [0, 1])$ be even such that $\tau \equiv 1$ in a neighbourhood of 0. We define

$$\xi_m = \sum_{k\in\mathbb{N}_0^2} \tau((mL)^{-1}k)\mathscr{W}(\mathfrak{n}_k)\mathfrak{n}_k,$$

where $\mathfrak{n}_k$ was extended by 0 beyond $[0,L]^2$.

Since we anticipate the requirement to renormalize the continuous Anderson Hamiltonian, we aim to define it as the limit of $\Delta + \xi_m - c_m$, where $(\xi_m)_{m\in\mathbb{N}}$ approximates the Neumann white noise ξ (in a way to be made precise below) and with $(c_m)_{m\in\mathbb{N}}$ being a particular diverging series of constants. This is implicit in the following definition:

Definition 1.21 [CvZ19, Definition 5.1]
We define the space of enhanced Neumann distributions $\mathfrak{X}_n^\alpha$ to be the closure in $\mathcal{C}_\mathfrak{n}^\alpha \times \mathcal{C}_\mathfrak{n}^{2\alpha+2}$ of

$$\{(\zeta, \zeta \odot \sigma(D)\zeta - c) | \zeta \in S_\mathfrak{n}([0,L]^2,\mathbb{R}), c \in \mathbb{R}\}.$$

We equip $\mathfrak{X}_n^\alpha$ with the relative topology in $\mathcal{C}_\mathfrak{n}^\alpha \times \mathcal{C}_\mathfrak{n}^{2\alpha+2}$ and the norm $\|\boldsymbol{\xi}\|_{\mathfrak{X}_\mathfrak{n}^\alpha}^2 = \|\xi\|_{\mathcal{C}_\mathfrak{n}^\alpha}^2 + \|\Xi\|_{\mathcal{C}_\mathfrak{n}^{2\alpha+2}}^2$ for $\boldsymbol{\xi} = (\xi, \Xi) \in \mathfrak{X}_\mathfrak{n}^\alpha$.

As it was already mentioned, we need to enhance white noise. The second component of the elements $\boldsymbol{\xi} \in \mathfrak{X}_\mathfrak{n}^\alpha$ corresponds to this enhancement. The main result of this section is the convergence and regularity of enhanced white noise:

Theorem 1.22 [CvZ19, Theorem 6.4]
Let $\alpha < -1$. There exists some random variable $\boldsymbol{\xi} \in \mathfrak{X}_\mathfrak{n}^\alpha$ such that almost surely in $\mathfrak{X}_\mathfrak{n}^\alpha$,

$$\lim_{m\to\infty} (\xi_m, \xi_m \odot \vartheta_m - c_m) \to \boldsymbol{\xi} =: (\xi, \Xi),$$

with $c_m := 1/(2\pi)\log(m) + c_\tau$, where c_τ only depends on τ. For $\phi, \psi \in S_\mathfrak{n}([0,L]^2,\mathbb{C})$, $\langle \xi, \phi\rangle$ and $\langle \xi, \psi\rangle$ are centred Gaussian random variables such that

$$\mathbb{E}(\langle \xi, \phi\rangle \overline{\langle \xi, \psi\rangle}^{\mathbb{C}}) = \langle \phi, \psi\rangle_{L^2([0,L]^2,\mathbb{C})}$$

and for any $\phi \in C_c^\infty([0,L]^2)$, $\langle \xi, \phi\rangle = W(\phi)$. The enhancement $\boldsymbol{\xi}$ is independent of τ.

The proof relies on Gaussian hypercontractivity, the Kolmogorov-Čentsov theorem and a careful analysis of the terms in the Littlewood-Paley decomposition.

Remark 1.23
The renormalizing sequence used here differs from the one stated in [CvZ19, Theorem 6.4] and will appear in an updated version of [CvZ19].

Let us also show a quantitative bound on the L^∞-norm of the mollified white noise as the mollification vanishes.

Lemma 1.24
It holds that a.s., $\|\xi_m\|_{L^\infty} \lesssim m^{1+\varepsilon}$ for any $\varepsilon > 0$.

Proof
It follows by Theorem 1.22 and linearity that $(\langle \xi, \mathfrak{n}_k \rangle)_{k\in\mathbb{N}_0^2}$ is multivariate normally distributed, which implies that $(\xi_m)_{m\in\mathbb{N}}$ has the same distribution as

$$\tau(m^{-1}D)\xi = \sum_{k\in\mathbb{N}_0^2} \tau((mL)^{-1}k) \langle \xi, \mathfrak{n}_k \rangle \mathfrak{n}_k.$$

We get for any $\varepsilon > 0$,

$$\|\tau(m^{-1}D)\xi\|_{L^\infty} \lesssim \|\tau(m^{-1}D)\xi\|_{\mathcal{C}_\mathfrak{n}^\varepsilon} = a_{2,\infty}\|\bar{\tau}(m^{-1}D)\bar{\xi}\|_{\mathcal{C}^\varepsilon},$$

where we used (5) in the last equality. By a version of the Hörmander-Mihlin inequality, [BCD11, Lemma 2.2], it follows that $\|\bar{\tau}(m^{-1}D)\bar{\xi}\|_{\mathcal{C}^\varepsilon} \lesssim m^{-\alpha+\varepsilon}\|\bar{\xi}\|_{\mathcal{C}^\alpha}$. Consequently for any $\varepsilon > 0$, $\|\tau(m^{-1}D)\xi\|_{L^\infty} \lesssim m^{-\alpha+\varepsilon}\|\xi\|_{\mathcal{C}_\mathfrak{n}^\alpha}$. Since $\alpha < -1$, the claim follows.

□

From here on we treat $\boldsymbol{\xi}_m = (\xi_m, \xi_m \odot \vartheta_m - c_m)$ and $\boldsymbol{\xi}$ realisation-wise and assume the regularity and convergence of Theorem 1.22.

1.4 Exposition: The Continuous Anderson Hamiltonian on the Square

In this section we define the continuous Anderson Hamiltonian with Dirichlet boundary conditions and characterize its spectrum. Most of the material can be found in [AC15] and [CvZ19, Sections 5, 6]. Let us stress that our construction of the killed rough Super Brownian Motion does not use the theory developed below: The reason is that the continuous Anderson Hamiltonian does in general not map into spaces of sufficiently high regularity. Nevertheless, its Eigenfunctions are linked to persistence properties, which we will consider in Section 4.

Let from here on $d = 2$, $\alpha \in (-4/3, -1)$, $\gamma \in (2/3, \alpha+2)$, $L > 0$. We also write $\mathcal{C}_\mathfrak{n}^\alpha([0,L]^2) =: \mathcal{C}_\mathfrak{n}^\alpha$, $H_0^\gamma([0,L]^2) =: H_0^\gamma$ and $L^2([0,L]^2) =: L^2$. We define $\sigma : \mathbb{R}^2 \to (0,\infty)$ by $x \mapsto \sigma(x) := (1+\pi^2|x|^2)^{-1}$.
We think of ξ as one realisation of Neumann white noise and define the domain of the continuous Anderson Hamiltonian.

Definition 1.25 [CvZ19, Definition 5.2]
Let $\boldsymbol{\xi} := (\xi, \Xi) \in \mathfrak{X}_\mathfrak{n}^\alpha$. We define the space of paracontrolled distributions $\mathscr{D}_\xi^{\mathfrak{d},\gamma} = \{f \in H_0^\gamma | f^\sharp := f - f \otimes \sigma(D)\xi \in H_0^{2\gamma}\}$. We define a scalar product on $\mathscr{D}_\xi^{\mathfrak{d},\gamma}$ by $\langle f, g\rangle_{\mathfrak{D}_\xi^{\mathfrak{d},\gamma}} = \langle f, g\rangle_{H_0^\gamma} + \langle f^\sharp, g^\sharp\rangle_{H_0^{2\gamma}}$.

We also define the subspace $\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d},\gamma}$ *of strongly paracontrolled distributions by* $\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d},\gamma} = \{f \in H_0^\gamma | f^\flat := f^\sharp - B(f,\boldsymbol{\xi}) \in H_0^2\}$*, where* $B(f,\boldsymbol{\xi}) = \sigma(D)(f\Xi + \xi \otimes f + \Delta f \otimes \sigma(D)\xi + 2\sum_{i=1}^d \partial_i f \otimes \partial_i \sigma(D)\xi)$*. We define a scalar product on* $\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d},\gamma}$ *by* $\langle f, g\rangle_{\mathcal{D}_{\boldsymbol{\xi}}^{\mathfrak{d},\gamma}} = \langle f, g\rangle_{H_0^\gamma} + \langle f^\flat, g^\flat\rangle_{H_0^2}$.

These definitions are partly motivated by Lemma 1.27 and Lemma 1.28 below. By modelling the theory around Eigenfunctions, one can guess the right paracontrolled structure by counting regularities. The details can be found in [AC15, above Definition 4.1, and Section 4.2].
Of course, $f^\sharp$ and $f^\flat$ depend on ξ and $\boldsymbol{\xi}$ respectively. That means that each enhanced Neumann distribution $\boldsymbol{\xi}$ yields a space of (strongly) paracontrolled distributions. We can now define the (Dirichlet) continuous Anderson Hamiltonian acting on such functions:

Definition 1.26 [CvZ19, Definition 5.3]
Let $\boldsymbol{\xi} = (\xi, \Xi) \in \mathfrak{X}_{\mathfrak{n}}^\alpha$*. We define formally the operator* $\mathscr{H}_{\boldsymbol{\xi}} : \mathscr{D}_{\xi}^{\mathfrak{d},\gamma} \to H_0^{\gamma-2}$ *by*

$$\mathscr{H}_{\boldsymbol{\xi}} f = \Delta f + \xi \otimes f + f \otimes \xi + f^\sharp \odot \xi + f\Xi + \mathscr{R}(f, \sigma(D)\xi, \xi),$$

where $\mathscr{R}$ *is the operator of Theorem 1.17. The terminology 'formally' refers to the requirement to verify that the operator does indeed map into* $H_0^{\gamma-2}$*.*

The next lemma removes the formal aspect of the previous definition and gives some first bounds.

Lemma 1.27
Let $\boldsymbol{\xi} = (\xi, \Xi) \in \mathfrak{X}_{\mathfrak{n}}^\alpha$*. Then it holds that* $\|\mathscr{H}_{\boldsymbol{\xi}} f\|_{H_0^{\gamma-2}} \lesssim \|f\|_{\mathscr{D}_{\xi}^{\mathfrak{d},\gamma}} (1 + \|\boldsymbol{\xi}\|_{\mathfrak{X}_n^\alpha})^2$.

Proof
Let $\boldsymbol{\xi} \in \mathfrak{X}_{\mathfrak{n}}^\alpha$ and $f \in \mathscr{D}_{\xi}^{\mathfrak{d},\gamma}$. We have

$$\mathscr{H}_{\boldsymbol{\xi}} f = \Delta f + \xi \otimes f + f \otimes \xi + f^\sharp \odot \xi + f\Xi + \mathscr{R}(f, \sigma(D)\xi, \xi).$$

Therefore by Bernstein's inequalities, Theorem 1.18, Theorem 1.13 and Theorem 1.19,

$$\begin{aligned}
&\|\Delta f\|_{H_0^{\gamma-2}} \lesssim \|f\|_{H_0^\gamma}, \quad \|\xi \otimes f\|_{H_0^{\gamma-2}} \lesssim \|f\|_{H_0^\gamma} \|\xi\|_{\mathcal{C}_{\mathfrak{n}}^\alpha},\\
&\|f \otimes \xi\|_{H_0^{\gamma-2}} \lesssim \|f\|_{H_0^\gamma} \|\xi\|_{\mathcal{C}_{\mathfrak{n}}^\alpha}, \quad \|f^\sharp \odot \xi\|_{H_0^{\gamma-2}} \lesssim \|f^\sharp\|_{H_0^{2\gamma}} \|\xi\|_{\mathcal{C}_{\mathfrak{n}}^\alpha},\\
&\|\mathscr{R}(f, \sigma(D)\xi, \xi)\|_{H_0^{\gamma-2}} \lesssim \|f\|_{H_0^\gamma} \|\xi\|_{\mathcal{C}_{\mathfrak{n}}^\alpha}^2, \quad \|f\Xi\|_{H_0^{\gamma-2}} \lesssim \|f\|_{H_0^\gamma} \|\Xi\|_{\mathcal{C}_{\mathfrak{n}}^{2\alpha+2}},
\end{aligned}$$

using that $\gamma - 2 < \alpha$, $\gamma - 2 < 2\gamma + \alpha$, $\gamma - 2 < \gamma + 2\alpha + 2$ and $\gamma - 2 < 2\alpha + 2$.

□

Lemma 1.28 [AC15, Proposition 4.20]
Let $\boldsymbol{\xi} = (\xi, \Xi) \in \mathfrak{X}_{\mathfrak{n}}^\alpha$*. If* $f \in \mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d},\gamma}$*, then* $\mathscr{H}_{\boldsymbol{\xi}} f \in L^2$*.*

Proof
If $f \in \mathfrak{D}^{\mathfrak{d},\gamma}_{\boldsymbol{\xi}}$, then it holds that $f = f \otimes \sigma(D)\xi + B(f,\boldsymbol{\xi}) + f^\flat$. So with $\Delta f = f - (1-\Delta)f$,

$$\begin{aligned}\mathscr{H}_{\boldsymbol{\xi}} f &= \Delta f + f \otimes \xi + \xi \otimes f + f^\sharp \odot \xi + \mathscr{R}(f,\sigma(D)\xi,\xi) + f\Xi \\ &= f - (-\Delta) f \otimes \sigma(D)\xi - f \otimes (1-\Delta)\sigma(D)\xi + 2\nabla f \otimes \nabla\sigma(D)\xi \\ &\quad - (1-\Delta)B(f,\boldsymbol{\xi}) - (1-\Delta)f^\flat + f \otimes \xi + \xi \otimes f + f^\sharp \odot \xi + \mathscr{R}(f,\sigma(D)\xi,\xi) + f\Xi \\ &= f - (1-\Delta)f^\flat + f^\sharp \odot \xi + \mathscr{R}(f,\sigma(D)\xi,\xi),\end{aligned}$$

where in the last line we used the definition of $B(f,\boldsymbol{\xi})$. We get for $\delta > 0$, $f \in H_0^\gamma$, $f^\flat \in H_0^2$, $\Delta f^\flat \in L^2$, $f^\sharp \odot \xi \in H_0^{2\gamma+\alpha}$, $\mathscr{R}(f,\sigma(D)\xi,\xi) \in H_0^{\gamma+2\alpha+2-\delta}$. For δ sufficiently small we get indeed the regularity L^2 using that $2\gamma+\alpha > 0$ and $\gamma+2\alpha+2 > 0$.

□

The next theorem concerns the spectrum of the continuous Anderson Hamiltonian.

Theorem 1.29 [CvZ19, Theorem 5.4]
Let $\boldsymbol{\xi} \in \mathfrak{X}^\alpha_{\mathfrak{n}}$. Then it holds that $\mathscr{H}_{\boldsymbol{\xi}}(\mathfrak{D}^{\mathfrak{d}}_{\boldsymbol{\xi}}) \subset L^2$ and $\mathscr{H}_{\boldsymbol{\xi}} : \mathfrak{D}^{\mathfrak{d},\gamma}_{\boldsymbol{\xi}} \to L^2$ is closed and self-adjoint as an operator on L^2. There exist $\lambda_1(\boldsymbol{\xi}) \geq \lambda_2(\boldsymbol{\xi}) \geq \lambda_3(\boldsymbol{\xi}) \geq ...$ such that the spectrum equals the point spectrum via $\mathscr{S}(\mathscr{H}_{\boldsymbol{\xi}}) = \mathscr{S}_p(\mathscr{H}_{\boldsymbol{\xi}}) = \{\lambda_n(\boldsymbol{\xi}) | n \in \mathbb{N}\}$ and Card$\{n \in \mathbb{N} | \lambda_n(\boldsymbol{\xi}) = \lambda\}$ = Dim Ker$(\lambda - \mathscr{H}_{\boldsymbol{\xi}}) < \infty$ for any $\lambda \in \mathscr{S}(\mathscr{H}_{\boldsymbol{\xi}})$. One has $L^2 = \overline{Span(\{u_k(\boldsymbol{\xi}) | k \in N\})}$ with $(u_k(\boldsymbol{\xi}))_{k\in\mathbb{N}}$ being the associated orthonormal system of Eigenfunctions of $\mathscr{H}_{\boldsymbol{\xi}}$. What is more, there exists some $N > 0$ such that for any $n \in \mathbb{N}$ and $\boldsymbol{\xi}, \boldsymbol{\theta} \in \mathfrak{X}^\alpha_{\mathfrak{n}}$, $|\lambda_n(\boldsymbol{\xi}) - \lambda_n(\boldsymbol{\theta})| \lesssim \|\boldsymbol{\xi} - \boldsymbol{\theta}\|_{\mathfrak{X}^\alpha_{\mathfrak{n}}}(1 + \|\boldsymbol{\xi}\|_{\mathfrak{X}^\alpha_{\mathfrak{n}}} + \|\boldsymbol{\theta}\|_{\mathfrak{X}^\alpha_{\mathfrak{n}}})^N$.

Finally we state the result that the largest Eigenvalue of $\mathscr{H}_{\boldsymbol{\xi}}$ will become positive almost surely as the size of the square increases. Along with the continuity of the Eigenvalues in Theorem 1.29, this will play a crucial role in Section 4.

Theorem 1.30 [CvZ19, Theorem 2.8]
Denote by $\lambda_1(\boldsymbol{\xi}, L)$ the largest Eigenvalue of $\mathscr{H}_{\boldsymbol{\xi}}$ on $[0,L]^2$. Then there exists some $\rho_1 > 0$ such that almost surely,

$$\lim_{L\to\infty, L=2^n, n\in\mathbb{N}} \frac{\lambda_1(\boldsymbol{\xi}, L)}{\log(L)} = \frac{2}{\rho_1}.$$

Note that for any $m \in \mathbb{N}$, $\mathfrak{D}^{\mathfrak{d},\gamma}_{\boldsymbol{\xi}_m} = H_0^2$ and $\mathscr{H}_{\boldsymbol{\xi}_m} = \Delta + \xi_m - c_m$ may be expressed as above, see [CvZ19, 5.9]. We will often use the notation $\mathscr{H}_{\boldsymbol{\xi}_m} := \Delta + \xi_m - c_m$. In particular, since $\boldsymbol{\xi}_m \to \boldsymbol{\xi}$ in $\mathfrak{X}^\alpha_{\mathfrak{n}}$, we get the continuity of the Eigenvalues of $\mathscr{H}_{\boldsymbol{\xi}_m}$ to $\mathscr{H}_{\boldsymbol{\xi}}$.

1.5 The Mollified Dirichlet Parabolic Anderson Model

In this section we begin the derivation of the solution theory for the Dirichlet Parabolic Anderson Model (PAM) by considering the mollified case. We follow the approach of

[GIP15, Section 5] with the differences that we consider the linear (non-generalized) model, include Dirichlet boundary conditions and use a slightly different enhancement of white noise. Let from here on $\boldsymbol{\xi}_m$ be a mollified Neumann white noise as constructed in Section 1.3.

First we introduce some parabolic Hölder spaces which are appropriate for our solution theory.

Definition 1.31 [GP17, Section 2.1]
Let $\beta \in (0,2)$. We define the space $\mathscr{L}^{\beta}_{\mathfrak{d},T} := C_T\mathcal{C}^{\beta}_{\mathfrak{d}} \cap C^{\beta/2}_T L^{\infty}$ equipped with the norm

$$\|u\|_{\mathscr{L}^{\beta}_{\mathfrak{d},T}} := \max\{\|u\|_{C_T\mathcal{C}^{\beta}_{\mathfrak{d}}}, \|u\|_{C^{\beta/2}_T L^{\infty}}\}.$$

One of the most important properties of this space is given by the following:

Lemma 1.32 [GP17, Lemma 2.11]
Let $\beta \in (0,2)$, $T > 0$ and let $f \in \mathscr{L}^{\beta}_{\mathfrak{d},T}$. Then for any $\delta \in (0,\beta]$, we have

$$\|f\|_{\mathscr{L}^{\delta}_{\mathfrak{d},T}} \lesssim \|f(0)\|_{\mathcal{C}^{\delta}_{\mathfrak{d}}} + T^{(\beta-\delta)/2}\|f\|_{\mathscr{L}^{\beta}_{\mathfrak{d},T}}.$$

Proof
The assertion for the periodic case can be found in [GP17, Lemma 2.11] and the extension to the case with Dirichlet boundary conditions follows by the definition via odd extensions.

□

The parabolic Hölder spaces also inherit the compact embedding properties from the non-parabolic ones.

Lemma 1.33
Let $0 < \beta' < \beta < 2$ be non-integer and $T > 0$. Then it holds that $\mathscr{L}^{\beta}_{\mathfrak{d},T}$ is compactly embedded in $\mathscr{L}^{\beta'}_{\mathfrak{d},T}$.

Proof
Let $(u_n)_{n\in\mathbb{N}}$ be bounded in $\mathscr{L}^{\beta}_{\mathfrak{d},T}$. It follows by the Arzelà-Ascoli theorem, that there exists some u, such that for a subsequence, $u_{n_k} \to u$ as $k \to \infty$ in $C_T L^{\infty}$. For this u, uniformly in $s \neq t \in [0,T]$,

$$\|u(t) - u(s)\|_{L^{\infty}} \leq \lim_{k\to\infty} \|u_{n_k}(t) - u_{n_k}(s)\|_{L^{\infty}} \leq C|t-s|^{\beta/2},$$

and uniformly in $t \in [0,T]$, for $x \neq y \in (0,L)^2$,

$$|u(t,x) - u(t,y)| \leq \lim_{k\to\infty} |u_{n_k}(t,x) - u_{n_k}(t,y)| \leq C|x-y|^{\beta}$$

for some constant $C > 0$. Therefore

$$\frac{\|(u-u_{n_k})(t)-(u-u_{n_k})(s)\|_{L^\infty}}{|t-s|^{\beta'/2}}$$
$$= \left(\frac{\|(u-u_{n_k})(t)-(u-u_{n_k})(s)\|_{L^\infty}}{|t-s|^{\beta/2}}\|(u-u_{n_k})(t)-(u-u_{n_k})(s)\|_{L^\infty}^{\frac{\beta}{\beta'}-1}\right)^{\frac{\beta'}{\beta}}.$$

The first factor is bounded uniformly in $s \neq t$ and the second vanishes. Similarly,

$$\frac{|(u-u_{n_k})(t,x)-(u-u_{n_k})(t,y)|}{|x-y|^{\beta'}}$$
$$= \left(\frac{|(u-u_{n_k})(t,x)-(u-u_{n_k})(t,y)|}{|x-y|^{\beta}}|(u-u_{n_k})(t,x)-(u-u_{n_k})(t,y)|^{\frac{\beta}{\beta'}-1}\right)^{\frac{\beta'}{\beta}}.$$

Again, the first factor is bounded uniformly in $t \in [0,T]$ and $x \neq y \in (0,L)^2$ and the second vanishes. This yields that $u_{n_k} \to u$ in $\mathscr{L}^{\beta'}_{\mathfrak{d},T}$, which is the claim.

□

We will also need to consider some explosive counterparts.

Definition 1.34 [GP17, Section 6]
Let $\gamma \geq 0$ and $\beta \in (0,2)$. We define $M^\gamma_T\mathcal{C}^\beta_\mathfrak{d} := \{v : [0,T] \to S'_\mathfrak{d}([0,L]^2) | \|v\|_{M^\gamma_T\mathcal{C}^\beta_\mathfrak{d}} < \infty\}$, where

$$\|v\|_{M^\gamma_T\mathcal{C}^\beta_\mathfrak{d}} = \sup_{t\in[0,T]}\left(t^\gamma\|v(t)\|_{\mathcal{C}^\beta_\mathfrak{d}}\right).$$

We define the space $\mathscr{L}^{\gamma,\beta}_{\mathfrak{d},T} := \{f : [0,T] \to S'_\mathfrak{d}([0,L]^2) | \|f\|_{\mathscr{L}^{\gamma,\beta}_{\mathfrak{d},T}} < \infty\}$, where

$$\|f\|_{\mathscr{L}^{\gamma,\beta}_{\mathfrak{d},T}} := \max\{\|t \mapsto t^\gamma f(t)\|_{C^{\beta/2}_T L^\infty}, \|f\|_{M^\gamma_T\mathcal{C}^\beta_\mathfrak{d}}\}.$$

Again, those spaces come with an important relation, which will allow us to obtain small scaling coefficients:

Lemma 1.35 [GP17, Lemma 6.8]
Let $\beta \in (0,2)$, $\gamma \in (0,1)$, $T > 0$ and let $f \in \mathscr{L}^{\gamma,\beta}_{\mathfrak{d},T}$. Then for any $\varepsilon \in [0, \beta \wedge 2\gamma]$,

$$\|f\|_{\mathscr{L}^{\gamma-\varepsilon/2,\beta-\varepsilon}_{\mathfrak{d},T}} \lesssim \|f\|_{\mathscr{L}^{\gamma,\beta}_{\mathfrak{d},T}}.$$

Proof
The assertion for the periodic case can be found in [GP17, Lemma 6.8]. The claim with Dirichlet boundary conditions follows by the definition.

□

We will use the following results to establish that certain functions lie in $\mathscr{L}_{\mathfrak{d},T}$:

Lemma 1.36 [GIP15, Lemma A.7]
Let $(P_t)_{t\geq 0}$ be the semigroup generated by the Dirichlet Laplacian on $[0,L]^2$. Then for any $T>0$, $\alpha\in\mathbb{R}$, $\delta\geq 0$ and $u\in S'_{\mathfrak{d}}([0,L]^2)$, uniformly in $t\in(0,T]$,

$$\|P_t u\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+\delta}} \lesssim_T t^{-\delta/2}\|u\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha}} \quad \textit{and} \quad \|P_t u\|_{\mathcal{C}_{\mathfrak{d}}^{\delta}} \lesssim_T t^{-\delta/2}\|u\|_{L^\infty}.$$

Proof
Let us momentarily denote by $(P_t^{\mathrm{Dir}})_{t\geq 0}$ the semigroup for the Dirichlet Laplacian and by $(P_t^{\mathrm{per}})_{t\geq 0}$ the semigroup for the periodic Laplacian. The Dirichlet semigroup is given by the formula

$$\widetilde{P_t^{\mathrm{Dir}}u} = \widetilde{\tau(\sqrt{t}D)u} = \tau(\sqrt{t}D)\tilde{u} = P_t^{\mathrm{per}}\tilde{u},$$

with $\tau(x)=\exp(-\pi^2|x|^2)$, where we have used (5). Indeed, if u is a proper function, then we can first start the heat equation on the torus from $\tilde{u}$, note that the solution retains its odd symmetry and then restrict to $[0,L]^2$. We get

$$\|P_t^{\mathrm{Dir}}u\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+\delta}} = a_{2,\infty}\|\widetilde{P_t^{\mathrm{Dir}}u}\|_{\mathcal{C}^{\alpha+\delta}} = a_{2,\infty}\|P_t^{\mathrm{per}}\tilde{u}\|_{\mathcal{C}^{\alpha+\delta}}.$$

The claim now follows from [GIP15, Lemma A.7].

□

Remark 1.37
Let $(P_t^{\mathrm{Neu}})_{t\geq 0}$ be the semigroup for the Neumann Laplacian. Then Lemma 1.36 carries over with the same line of arguments.

Lemma 1.38 [GIP15, Lemma A.8]
Let $(P_t)_{t\geq 0}$ be the semigroup for the Dirichlet Laplacian on $[0,L]^2$. Let $\beta\in(0,1)$ and $u\in\mathcal{C}_{\mathfrak{d}}^{\beta}$. Then we have for all $t\geq 0$,

$$\|(P_t-1)u\|_{L^\infty} \lesssim t^{\beta/2}\|u\|_{\mathcal{C}_{\mathfrak{d}}^{\beta}}.$$

Proof
The proof follows by [GIP15, Lemma A.8] and the reasoning of the proof of Lemma 1.36.

□

A generalization of Lemma 1.38 to different exponents is given by the following Schauder estimates:

Lemma 1.39 [GP17, Lemma 2.9]
Let $(P_t)_{t\geq 0}$ be the semigroup for the Dirichlet Laplacian on $[0,L]^2$. We define for $f\in$

$C([0,\infty), S'_{\mathfrak{d}}([0,L]^2))$, $If(t) := \int_0^t P_{t-s} f(s) ds$. *Let* $\beta \in (0,2)$, *then for any* $T > 0$,

$$\|If\|_{\mathscr{L}^{\beta}_{\mathfrak{d},T}} \lesssim \|f\|_{C_T \mathcal{C}^{\beta-2}_{\mathfrak{d}}} \quad \text{and} \quad \|Pu_0\|_{\mathscr{L}^{\beta}_{\mathfrak{d},T}} \lesssim \|u_0\|_{\mathcal{C}^{\beta}_{\mathfrak{d}}}.$$

Finally we also need the following analogue for the explosive spaces:

Lemma 1.40 [GP17, Lemma 6.6]
Let $\beta \in (0,2)$, $T > 0$ *and* $\gamma \in [0,1)$. *Then*

$$\|If\|_{\mathscr{L}^{\gamma,\beta}_{\mathfrak{d},T}} \lesssim \|f\|_{M^{\gamma}_T \mathcal{C}^{\beta-2}_{\mathfrak{d}}} \quad \text{and} \quad \text{if } \delta \geq -\beta, \text{ then } \|Pu_0\|_{\mathscr{L}^{(\beta+\delta)/2,\beta}_{\mathfrak{d},T}} \lesssim \|u_0\|_{\mathcal{C}^{-\delta}_{\mathfrak{d}}}.$$

For any $\beta \in \mathbb{R}$, $\gamma \in [0,1)$ *and* $T > 0$,

$$\|If\|_{M^{\gamma}_T \mathcal{C}^{\beta}_{\mathfrak{d}}} \lesssim \|f\|_{M^{\gamma}_T \mathcal{C}^{\beta-2}_{\mathfrak{d}}}.$$

Now for the solution theory of the PAM, we first consider a sequence of mollified white noises $(\boldsymbol{\xi}_m)_{m\in\mathbb{N}}$ which we assume converge to $\boldsymbol{\xi}$ in $\mathfrak{X}^{\alpha+4\varepsilon}_{\mathfrak{n}}$, with $\varepsilon \in (0, 1/4(-1-\alpha))$.[1] From here on, we denote $L = \partial_t - \Delta$ and recall that $\mathscr{H}_{\boldsymbol{\xi}_m} = \Delta + \xi_m - c_m$. We first show that several notions of solutions to the mollified PAM found in the literature are in fact compatible.

Theorem 1.41
Let $m \in \mathbb{N}$, $u_0^m \in \mathcal{C}^{\alpha+2}_{\mathfrak{d}}$ *and* $T > 0$. *Then there exists a unique, uniformly continuous solution* $u_m \in \mathscr{L}^{\alpha+2}_{\mathfrak{d},T} \cap C^{1,2}((0,T)\times(0,L)^2)$ *to the mollified Dirichlet Parabolic Anderson Model given by*

$$\begin{cases} Lu_m = (\xi_m - c_m)u_m \quad \text{in } (0,T)\times(0,L)^2, \\ u_m(0) = u_0^m \ \text{in } [0,L]^2, \ u_m = 0 \ \text{on } [0,T]\times\partial[0,L]^2. \end{cases}$$

We define

$$Dom(\mathscr{H}_{\boldsymbol{\xi}_m}) := \{f | f \in C_0((0,L)^2), \mathscr{H}_{\boldsymbol{\xi}_m} f \in C_0((0,L)^2)\}.$$

There exists a semigroup associated to $\mathscr{H}_{\boldsymbol{\xi}_m}$ *and if* $u_0^m \in Dom(\mathscr{H}_{\boldsymbol{\xi}_m})$, *then there exists a unique mild and also strong solution* $u_m \in C^1([0,\infty), C_0((0,L)^2))$ *in this sense. The solution satisfies for any* $0 \leq t < \infty$, $u_m(t) \in Dom(\mathscr{H}_{\boldsymbol{\xi}_m})$. *We call those notions of solutions of Lieberman- or respectively semigroup-sense with respect to* $\mathscr{H}_{\boldsymbol{\xi}_m}$. *Both notions of solutions are consistent with one another.*

[1] The reason why we need the noise to lie in a slightly better space will become clear below. Since the regularity $\alpha < -1$ of white noise is not sharp, we may assume that it actually lies in this space.

Proof
The first claim follows from [Lie89, Theorem 13.3] after noting that $(0,T) \times (0,L)^2$ satisfies a uniform exterior tusk condition on its parabolic boundary. Also, following the observations below [Lie89, Theorem 13.3], using that the 'width' R_0 of the tusk may be chosen arbitrarily large, we may assume $\sigma_0 = 1$; all of the above in the notation of [Lie89, Theorem 13.1]. Therefore we may choose, again in this notation, $\sigma = \alpha + 2$, which yields the claim.
For the second claim assume that $u_0^m \in \mathrm{Dom}(\mathscr{H}_{\boldsymbol{\xi}_m})$. Note that by [CZ95, Theorem 3.17, Proposition 3.23 and above], there exists a strongly continuous semigroup on $C_0((0,L)^2)$ associated to $\mathscr{H}_{\boldsymbol{\xi}_m}$, which yields a mild and also strong solution in $\mathrm{Dom}(\mathscr{H}_{\boldsymbol{\xi}_m})$.
Let v_m be a solution of semigroup-sense started from $u_0^m \in \mathrm{Dom}(\mathscr{H}_{\boldsymbol{\xi}_m}) \cap \mathcal{C}_{\mathfrak{d}}^{\alpha+2}$. We will see below that $v_m = Pu_0^m + I((\xi_m - c_m)v_m)$. By Lemma 1.39, $\|Pu_0^m\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} \lesssim \|u_0^m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}$. Next by Lemma 1.40, $\|I((\xi_m - c_m)v_m)\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} \lesssim \|(\xi_m - c_m)v_m\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha}} \lesssim \|(\xi_m - c_m)v_m\|_{C_T L^\infty}$. Therefore, $v_m \in C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}$. We iterate again and arrive at the bound $\|I((\xi_m - c_m)v_m)\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+4}} \lesssim \|(\xi_m - c_m)v_m\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}$. By Lemma 1.36 for any $t > 0$, $P_t u_0^m \in C^2((0,L)^2)$. This yields for any $t > 0$, $v_m(t) \in C^2((0,L)^2)$, since $\alpha + 4 > 2$.
Let now u be a solution of Lieberman-sense. In order to establish that $u = v$, it suffices to apply the maximum principle [Eva10, Theorem 7.1.4.9] to $w(t) := \exp(-At)(u(t) - v(t))$, where $A := \sup_{x\in(0,L)^2} |\xi_m(x) - c_m|$.

□

Definition 1.42
Note that Dom($\mathscr{H}_{\boldsymbol{\xi}_m}$) is independent of m and coincides with the domain of the Dirichlet Laplacian on $(0,L)^2$. In particular,

$$Dom(\mathscr{H}_{\boldsymbol{\xi}_m}) = \{f | f \in C_0((0,L)^2), \Delta f \in C_0((0,L)^2)\}.$$

We define $I_{PAM}^m := Dom(\mathscr{H}_{\boldsymbol{\xi}_m}) \cap \mathcal{C}_{\mathfrak{d}}^{\alpha+2}$.

We finally need to define a modified paraproduct to include some smoothing in time. This will allow us to let it commute with L modulo more regular terms.

Definition 1.43 [GIP15, Section 5][GP17, Section 2.3]
Let $\phi \in C_c^\infty(\mathbb{R})$ be non-negative, of total mass 1, and assume that Supp(ϕ) $\subset (0,\infty)$. We define for any $i \geq -1$, $Q_i : C_T S'_{\mathfrak{d}}([0,L]^2) \to C_T S'_{\mathfrak{d}}([0,L]^2)$ by $Q_i f(t) = \int_{\mathbb{R}} 2^{2i}\phi(2^{2i}(t - s)) f((s \wedge T) \vee 0) ds$. We define a modified paraproduct by

$$f \varolessthan_T g = \sum_{i\geq -1} \sum_{j=-1}^{i-2} \Delta_j Q_i f \Delta_i g, \quad f \in C_T S'_{\mathfrak{d}}([0,L]^2),\ g \in C_T S'_{\mathfrak{n}}([0,L]^2).$$

Note that if g is independent of time, then it follows that $(f \varolessthan_T g)(0) = f(0) \varolessthan g$, since $\mathrm{Supp}(\phi) \subset (0,\infty)$ and ϕ is of mass 1. Some crucial properties of the modified paraproduct are summarized in the following:

Lemma 1.44 [GIP15, Lemma 5.1 and above]
Let $T > 0$, $\delta \in (0,1)$, $\beta \in \mathbb{R}$. Let $f \in C_T S'_{\mathfrak{d}}([0,L]^2)$ and $g : [0,T] \to \mathcal{C}^{\beta}_{\mathfrak{n}}$. Then it holds that

$$\|(f \otimes_T g)(t)\|_{\mathcal{C}^{\beta}_{\mathfrak{d}}} \lesssim \|f\|_{C_T L^\infty} \|g(t)\|_{\mathcal{C}^{\beta}_{\mathfrak{n}}}.$$

Let $u \in \mathscr{L}^{\delta}_{\mathfrak{d},T}$ and $v \in C_T \mathcal{C}^{\beta}_{\mathfrak{n}}$. Then

$$\|L(u \otimes_T v) - u \otimes_T Lv\|_{C_T \mathcal{C}^{\delta+\beta-2}_{\mathfrak{d}}} \lesssim \|u\|_{\mathscr{L}^{\delta}_{\mathfrak{d},T}} \|v\|_{C_T \mathcal{C}^{\beta}_{\mathfrak{n}}}$$

as well as

$$\|u \otimes v - u \otimes_T v\|_{C_T \mathcal{C}^{\delta+\beta}_{\mathfrak{d}}} \lesssim \|u\|_{C_T^{\delta/2} L^\infty} \|v\|_{C_T \mathcal{C}^{\beta}_{\mathfrak{n}}}.$$

Proof
By expressing Δ as a Fourier multiplier and using (5), we get for any $u \in S'_{\mathfrak{d}}([0,L]^2)$, $\widetilde{\Delta u} = \Delta \tilde{u}$. Therefore the results follow from [GIP15, Lemma 5.1 and above] using the definition of $\mathcal{C}_{\mathfrak{d}}$ and $\mathcal{C}_{\mathfrak{n}}$, as the mollification in time will not affect symmetries in space.
□

Those are all the theoretical results we need. Now we can focus on the construction of the Dirichlet Parabolic Anderson Model.
The goal is to first establish natural bounds for the solutions associated to the mollified noises and then to establish the local Lipschitz continuity of the solution map.
By parabolic regularity we expect the appropriate space for the solution to the Parabolic Anderson Model to be $\mathscr{L}^{\alpha+2}_{\mathfrak{d},T}$. Let $\vartheta_m = (1-\Delta)^{-1}\xi_m$. Recall that $\alpha \in (-4/3, -1)$. In order to decompose the ill-defined product $u\xi$, we make the paracontrolled Ansatz $u = u \otimes_T \vartheta + u^\sharp$, with $u^\sharp \in \mathcal{C}^{2(\alpha+2)}_{\mathfrak{d}}$. Then $u^\sharp \xi$ is well-defined by Theorem 1.16, since $2\alpha + 4 + \alpha > 0$. Using the equation for u, we can deduce that the remainder $u^\sharp$ is given as a solution to a PDE as well:

Lemma 1.45
Let $u_0^m \in I^m_{PAM}$, $m \in \mathbb{N}$. Then u_m solves the PDE

$$\begin{cases} Lu_m = (\xi_m - c_m)u_m & \text{in } (0,T) \times (0,L)^2, \\ u_m(0) = u_0^m \text{ in } [0,L]^2, \ u_m = 0 \text{ on } [0,T] \times \partial[0,L]^2, \end{cases}$$

if and only if $u_m = u_m \otimes_T \vartheta_m + u_m^\sharp$ and $u_m^\sharp$ solves

$$\begin{cases} Lu_m^\sharp = \Phi_m^\sharp & \text{in } (0,T) \times (0,L)^2, \\ u_m^\sharp(0) = u_0^m - u_0^m \otimes \vartheta_m \text{ in } [0,L]^2, \ u_m^\sharp = 0 \text{ on } [0,T] \times \partial[0,L]^2. \end{cases} \tag{6}$$

The function $\Phi^\sharp_m$ is given by

$$\begin{aligned}\Phi^\sharp_m &= (u_m \otimes_T L\vartheta_m - L(u_m \otimes_T \vartheta_m)) + (u_m \otimes \xi_m - u_m \otimes L\vartheta_m)\\ &\quad + (u_m \otimes L\vartheta_m - u_m \otimes_T L\vartheta_m) + \xi_m \otimes u_m + \mathscr{R}(u_m, \xi_m, \vartheta_m) + u_m(\xi_m \odot \vartheta_m - c_m)\\ &\quad + u^\sharp_m \odot \xi_m + (u_m \otimes_T \vartheta_m - u_m \otimes \vartheta_m) \odot \xi_m.\end{aligned}$$

Note that u_m is a solution in both strong senses discussed in Lemma 1.41. First, we want to establish that u_m can also be seen as a mild solution of semigroup sense with respect to Δ. We fix u_m to be the strong solution to the mollified PAM and consider the equation for v_m,

$$\begin{cases} Lv_m = (\xi_m - c_m)u_m \quad \text{in } (0,T) \times (0,L)^2, \\ v_m(0) = u_0^m \text{ in } [0,L]^2, \; v_m = 0 \text{ on } [0,T] \times \partial[0,L]^2. \end{cases}$$

The function $u_m \in C([0,T], C_0((0,L)^2)) \cap C^1((0,T), C_0((0,L)^2))$ is a pointwise solution to this equation such that $u_m \in \mathrm{Dom}(\mathscr{H}_{\boldsymbol{\xi}_m})$. Consequently, it is a strong solution in the semigroup sense with respect to Δ, hence also a mild solution.
We have $u^\sharp_m \in C_0((0,L)^2)$, since $u_m \in C_0((0,L)^2)$ and $u_m \otimes_T \vartheta_m \in C_0((0,L)^2)$. Next note that

$$\Delta u^\sharp_m = \Delta u_m - \Delta u_m \otimes_T \vartheta_m - u_m \otimes_T \Delta\vartheta_m - 2\nabla u_m \otimes_T \nabla\vartheta_m.$$

We have $\Delta u_m \in C_0((0,L)^2)$, since $u_m \in \mathrm{Dom}(\mathscr{H}_{\boldsymbol{\xi}_m})$. What is more, $\Delta u_m \otimes_T \vartheta_m \in C_0((0,L)^2)$ and $u_m \otimes_T \Delta\vartheta_m \in C_0((0,L)^2)$ by Lemma 1.44 using that Δ does not change boundary conditions. Finally, the most tricky term, $\nabla u_m \otimes_T \nabla\vartheta_m = \sum_{i=1}^2 \partial_i u_m \otimes_T \partial_i\vartheta_m$, needs to be treated by an extension of the methods of [CvZ19, Section 4] to mixed boundary conditions, which is implicitly assumed in [CvZ19, Definition 5.2]. We therefore get $\Delta u^\sharp_m \in C_0((0,L)^2)$. Then $\partial_t u^\sharp_m = \Delta u^\sharp_m + \Phi^\sharp_m \in C_0((0,L)^2)$, since also $\Phi^\sharp_m \in C_0((0,L)^2)$ by what we have already established.
Consequently, $u^\sharp_m$ is a strong solution to the PDE (6) in the sense of semigroup theory with respect to Δ. This yields the variation-of-constants representations

$$\begin{aligned} u_m(t) &= P_t u_0^m + \int_0^t P_{t-s}((\xi_m - c_m)u_m(s))ds,\\ u^\sharp_m(t) &= P_t u^\sharp_m(0) + \int_0^t P_{t-s}\Phi^\sharp_m(s)ds. \end{aligned}$$

Hence, finally for the paraproduct term,

$$(u_m \otimes_T \vartheta_m)(t) = P_t(u_0^m \otimes \vartheta_m) + \int_0^t P_{t-s}L(u_m \otimes_T \vartheta_m)(s)ds.$$

There is one technical subtlety that we need to discuss: The commutator $\mathscr{R}$ was only defined implicitly beyond smooth functions. Hence, we need to show that the identity

$\mathscr{R}(u_m, \xi_m, \vartheta_m) = (u_m \otimes \vartheta_m) \odot \xi_m - u_m(\vartheta_m \odot \xi_m)$ also holds for $u_m \in \mathcal{C}_{\mathfrak{d}}^{\alpha+2}$. We have seen in the proof of Lemma 1.17, that one can approximate u_m in $\mathcal{C}_{\mathfrak{d}}^{\alpha+2-\delta}$, $0 < \delta < \alpha + 2$, by $u_m^k \in S_{\mathfrak{d}}([0, L]^2)$. Fortunately, the noise is still smooth, so we get for $\delta < -(2\alpha + 2)$ the (non-natural) bounds

$$\|\mathscr{R}(u_m^k - u_m, \xi_m, \vartheta_m)\|_{\mathcal{C}_{\mathfrak{d}}^{3\alpha+4}} \lesssim \|u_m^k - u_m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2-\delta}} \|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha}} \|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2+\delta}},$$

and

$$\|((u_m^k - u_m) \otimes \vartheta_m) \odot \xi_m\|_{\mathcal{C}_{\mathfrak{d}}^{3\alpha+4}} \lesssim \|u_m^k - u_m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2-\delta}} \|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{2\alpha+4}} \|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha}},$$
$$\|(u_m^k - u_m)(\vartheta_m \odot \xi_m)\|_{\mathcal{C}_{\mathfrak{d}}^{3\alpha+4}} \lesssim \|u_m^k - u_m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2-\delta}} \|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha}} \|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{2\alpha+4}}.$$

Letting $k \to \infty$ yields that $\mathscr{R}(u_m, \xi_m, \vartheta_m) = (u_m \otimes \vartheta_m) \odot \xi_m - u_m(\vartheta_m \odot \xi_m)$.
Using the various bounds established above, we arrive at the following bound for $\Phi_m^\sharp$:

$$\begin{aligned}
&\sup_{0 \le t \le T} \left(t^{(\alpha+2)/2} \|\Phi_m^\sharp(t)\|_{\mathcal{C}_{\mathfrak{d}}^{2(\alpha+2)-2}} \right) \\
&\lesssim T^{(\alpha+2)/2} \|u_m\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} \left(\|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2}} + \|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha}} + \|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha}} \|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2}} \right) \\
&+ T^{(\alpha+2)/2} \|u_m\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} \|\vartheta_m \odot \xi_m - c_m\|_{\mathcal{C}_{\mathfrak{n}}^{2\alpha+2}} \\
&+ T^{\varepsilon/2} \sup_{t \in [0,T]} \left(t^{(\alpha+2-\varepsilon)/2} \|u_m^\sharp(t)\|_{\mathcal{C}_{\mathfrak{d}}^{2(\alpha+2)-4\varepsilon}} \right) \|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+4\varepsilon}},
\end{aligned} \tag{7}$$

where $\varepsilon \in (0, 1/4(-1-\alpha))$. We stress that we used that the regularity of white noise is not sharp, i.e. we may assume that ξ_m actually lies in a space of higher regularity. This allows us to acquire a bound in terms of u_m in a space of lower regularity. Also note that ϑ_m was chosen such that $L\vartheta_m = (\partial_t - \Delta)\vartheta_m = \xi_m - \vartheta_m$. This was used such that terms of lesser regularity cancel in $\Phi_m^\sharp$ and consequently that $u_m^\sharp$ is of the prescribed regularity.

Step 1: The $\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}$-regularity of u_m.
We use the decomposition

$$u_m = u_m^\sharp + u_m \otimes_T \vartheta_m = u_m^\sharp + P(u_0^m \otimes \vartheta_m) + IL(u_m \otimes_T \vartheta_m).$$

By Lemma 1.44,

$$\begin{aligned}
\|L(u_m \otimes_T \vartheta_m)\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha}} &\le \|L(u_m \otimes_T \vartheta_m) - u_m \otimes_T L\vartheta_m\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha}} + \|u_m \otimes_T (\xi_m - \vartheta_m)\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha}} \\
&\lesssim \|u_m\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2-\varepsilon}} \|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2+\varepsilon}} + \|u_m\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2-\varepsilon}} (\|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+\varepsilon}} + \|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2+\varepsilon}}).
\end{aligned}$$

We get by Lemma 1.39 and Lemma 1.32,

$$\begin{aligned}
\|u_m\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} &\lesssim \|u_m^\sharp\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} + \|u_0^m \otimes \vartheta_m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} + \|L(u_m \otimes \vartheta_m)\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha}} \\
&\lesssim \|u_m^\sharp\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} + \|u_0^m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} \|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2}} \\
&+ \|u_m\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2-\varepsilon}} \|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2+\varepsilon}} + \|u_m\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2-\varepsilon}} (\|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+\varepsilon}} + \|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2+\varepsilon}}) \\
&\lesssim \|u_m^\sharp\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} + \|u_0^m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} (\|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+\varepsilon}} + \|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2+\varepsilon}}),
\end{aligned}$$

where in the last inequality we absorbed some terms for T sufficiently small.
Step 2: The $\mathscr{L}^{\alpha+2}_{\mathfrak{d},T}$-regularity of $u^\sharp_m$.
We proceed as in [GIP15, Section 5] and use the decomposition $u^\sharp_m = Pu^\sharp_m(0) + I\Phi^\sharp_m$.
We get by Lemma 1.39,

$$\|Pu^\sharp_m(0)\|_{\mathscr{L}^{\alpha+2}_{\mathfrak{d},T}} \lesssim \|u^\sharp_m(0)\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{d}}} \lesssim \|u^m_0\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{d}}}(1+\|\vartheta_m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{n}}}).$$

Next by Lemma 1.36,

$$\begin{aligned}&\left\|\int_0^t P_{t-s}\Phi^\sharp_m(s)ds\right\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{d}}} \leq \int_0^t \|P_{t-s}\Phi^\sharp_m(s)\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{d}}}ds \lesssim \int_0^t (t-s)^{(\alpha+2)/2-1}\|\Phi^\sharp_m(s)\|_{\mathcal{C}^{2(\alpha+2)-2}_{\mathfrak{d}}}ds\\ &\leq \int_0^t (t-s)^{(\alpha+2)/2-1}s^{-(\alpha+2)/2}ds \sup_{0\leq s\leq T}\left(s^{(\alpha+2)/2}\|\Phi^\sharp_m(s)\|_{\mathcal{C}^{2(\alpha+2)-2}_{\mathfrak{d}}}\right).\end{aligned}$$

Note that for the singularity,

$$\int_0^t (t-s)^{(\alpha+2)/2-1}s^{-(\alpha+2)/2}ds = \int_0^1 t^{(\alpha+2)/2-1}(1-s)^{(\alpha+2)/2-1}t^{-(\alpha+2)/2}s^{-(\alpha+2)/2}tds \lesssim 1.$$

Next for the temporal Hölder-regularity of the integral, we decompose it as

$$\begin{aligned}&\int_0^t P_{t-r}\Phi^\sharp_m(r)dr - \int_0^s P_{s-r}\Phi^\sharp_m(r)dr\\ &= \int_s^t P_{t-r}\Phi^\sharp_m(r)dr + \int_0^s (P_{t-s}-1)P_{s-r}\Phi^\sharp_m(r)dr.\end{aligned}$$

We compute for the first term with Lemma 1.36,

$$\begin{aligned}&\left\|\int_s^t P_{t-r}\Phi^\sharp_m(r)dr\right\|_{L^\infty} \leq \int_s^t \|P_{t-r}\Phi^\sharp_m(r)\|_{L^\infty}dr\\ &\lesssim \lim_{\delta\downarrow 0}\int_s^t \|P_{t-r}\Phi^\sharp_m(r)\|_{\mathcal{C}^\delta_{\mathfrak{d}}}dr \lesssim \lim_{\delta\downarrow 0}\int_s^t (t-r)^{(\alpha+2)-1-\delta/2}\|\Phi^\sharp_m(r)\|_{\mathcal{C}^{2(\alpha+2)-2}_{\mathfrak{d}}}dr\\ &\lesssim (t-s)^{(\alpha+2)/2}\lim_{\delta\downarrow 0}\int_s^t (t-r)^{(\alpha+2)/2-1-\delta/2}r^{-(\alpha+2)/2}dr\\ &\times \sup_{0\leq r\leq T}\left(r^{(\alpha+2)/2}\|\Phi^\sharp_m(r)\|_{\mathcal{C}^{2(\alpha+2)-2}_{\mathfrak{d}}}\right).\end{aligned}$$

For the singularity,

$$\int_s^t (t-r)^{(\alpha+2)/2-1-\delta/2}r^{-(\alpha+2)/2}dr \leq t^{-\delta/2}\int_0^1 (1-r)^{(\alpha+2)/2-1-\delta/2}r^{-(\alpha+2)/2}dr,$$

which converges as $\delta \downarrow 0$ by the dominated convergence theorem. For the second term it follows by Lemma 1.38 and Lemma 1.36,

$$\begin{aligned}
&\left\| \int_0^s (P_{t-s}-1)P_{s-r}\Phi_m^\sharp(r)dr \right\|_{L^\infty} \\
&\leq \int_0^s \|(P_{t-s}-1)P_{s-r}\Phi_m^\sharp(r)\|_{L^\infty} dr \lesssim \int_0^s (t-s)^{(\alpha+2)/2} \|P_{s-r}\Phi_m^\sharp(r)\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} dr \\
&\lesssim \int_0^s (t-s)^{(\alpha+2)/2}(s-r)^{(\alpha+2)/2-1} \|\Phi_m^\sharp(r)\|_{\mathcal{C}_{\mathfrak{d}}^{2(\alpha+2)-2}} dr \\
&\leq (t-s)^{(\alpha+2)/2} \int_0^s (s-r)^{(\alpha+2)/2-1} r^{-(\alpha+2)/2} dr \sup_{0\leq r\leq T} \left(r^{(\alpha+2)/2} \|\Phi_m^\sharp(r)\|_{\mathcal{C}_{\mathfrak{d}}^{2(\alpha+2)-2}} \right).
\end{aligned}$$

The singularity is the same as before. Consequently, we get that

$$\|u_m^\sharp\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} \lesssim \|u_0^m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}(1+\|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2}}) + \sup_{0\leq t\leq T} \left(t^{(\alpha+2)/2} \|\Phi_m^\sharp(t)\|_{\mathcal{C}_{\mathfrak{d}}^{2(\alpha+2)-2}} \right).$$

Step 3: The $\mathscr{L}_{\mathfrak{d},T}^{(\alpha+2+\varepsilon)/2,2(\alpha+2)-2\varepsilon}$-regularity of $u_m^\sharp$.
We use the decomposition $u_m^\sharp = Pu_m^\sharp(0) + I\Phi_m^\sharp$. By Lemma 1.40,

$$\|Pu_m^\sharp(0)\|_{\mathscr{L}_{\mathfrak{d},T}^{(\alpha+2+\varepsilon)/2,2(\alpha+2)-2\varepsilon}} \lesssim \|u_m^\sharp(0)\|_{\mathcal{C}_{\mathfrak{d}}^{(\alpha+2)-3\varepsilon}} \lesssim \|u_0^m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}(1+\|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2}}),$$

where we chose in the notation of the lemma, $\beta = 2(\alpha+2)-2\varepsilon$ and $\delta = -(\alpha+2)+3\varepsilon$. Next, again by Lemma 1.40,

$$\|I\Phi_m^\sharp\|_{\mathscr{L}_{\mathfrak{d},T}^{(\alpha+2+\varepsilon)/2,2(\alpha+2)-2\varepsilon}} \lesssim \|\Phi_m^\sharp\|_{M_T^{(\alpha+2+\varepsilon)/2}\mathcal{C}_{\mathfrak{d}}^{2(\alpha+2)-2-2\varepsilon}}.$$

All in all this yields

$$\begin{aligned}
\|u_m^\sharp\|_{\mathscr{L}_{\mathfrak{d},T}^{(\alpha+2+\varepsilon)/2,2(\alpha+2)-2\varepsilon}} &\lesssim \|u_0^m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}(1+\|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2}}) \\
&\quad + T^{\varepsilon/2} \sup_{0\leq t\leq T} \left(t^{(\alpha+2)/2} \|\Phi_m^\sharp(t)\|_{\mathcal{C}_{\mathfrak{d}}^{2(\alpha+2)-2}} \right).
\end{aligned}$$

Step 4: Closing the bounds.
In this step we derive closed forms of the bounds above. We have by Steps 1 and 2,

$$\begin{aligned}
&\|u_m\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} \lesssim \|u_m^\sharp\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} + \|u_0^m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}(\|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+\varepsilon}} + \|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2+\varepsilon}}) \\
&\lesssim \|u_0^m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}(\|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2+\varepsilon}} + \|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+\varepsilon}} + 1) + \sup_{0\leq t\leq T} \left(t^{(\alpha+2)/2} \|\Phi_m^\sharp(t)\|_{\mathcal{C}_{\mathfrak{d}}^{2(\alpha+2)-2}} \right).
\end{aligned}$$

By plugging this into (7), absorbing some terms for T sufficiently small and applying Lemma 1.35,

$$\begin{aligned}
&\sup_{0\leq t\leq T} \left(t^{(\alpha+2)/2} \|\Phi_m^\sharp(t)\|_{\mathcal{C}_{\mathfrak{d}}^{2(\alpha+2)-2}} \right) \\
&\lesssim T^{(\alpha+2)/2} \|u_0^m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}(\|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2+\varepsilon}} + \|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+\varepsilon}} + 1)(\|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2}} + \|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha}} \\
&+ \|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha}}\|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2}} + \|\vartheta_m \odot \xi_m - c_m\|_{\mathcal{C}_{\mathfrak{n}}^{2\alpha+2}}) + T^{\varepsilon/2}\|u_m^\sharp\|_{\mathscr{L}_{\mathfrak{d},T}^{(\alpha+2+\varepsilon)/2,2(\alpha+2)-2\varepsilon}} \|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+4\varepsilon}}.
\end{aligned}$$

In the application of Lemma 1.35, we used the coefficients $\beta = 2(\alpha+2) - 2\varepsilon$, $\gamma = (\alpha+2+\varepsilon)/2$ and $\varepsilon' = 2\varepsilon$. By applying the bound of Step 3 and absorbing one more term for T sufficiently small,

$$\begin{aligned}&\sup_{0\le t\le T}\left(t^{(\alpha+2)/2}\|\Phi^\sharp_m(t)\|_{\mathcal{C}^{2(\alpha+2)-2}_{\mathfrak{d}}}\right)\\ &\lesssim T^{(\alpha+2)/2}\|u_0^m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{d}}}(\|\vartheta_m\|_{\mathcal{C}^{\alpha+2+\varepsilon}_{\mathfrak{n}}} + \|\xi_m\|_{\mathcal{C}^{\alpha+\varepsilon}_{\mathfrak{n}}} + 1)(\|\vartheta_m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{n}}} + \|\xi_m\|_{\mathcal{C}^{\alpha}_{\mathfrak{n}}}\\ &+ \|\xi_m\|_{\mathcal{C}^{\alpha}_{\mathfrak{n}}}\|\vartheta_m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{n}}} + \|\vartheta_m\odot\xi_m - c_m\|_{\mathcal{C}^{2\alpha+2}_{\mathfrak{n}}}) + T^{\varepsilon/2}\|u_0^m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{d}}}(1+\|\vartheta_m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{n}}})\|\xi_m\|_{\mathcal{C}^{\alpha+4\varepsilon}_{\mathfrak{n}}}.\end{aligned}$$

By plugging this back into the bound for $\|u_m\|_{\mathscr{L}^{\alpha+2}_{\mathfrak{d},T}}$, we get the following:

Lemma 1.46
With the notation of Lemma 1.45, we get the bound for T sufficiently small:

$$\begin{aligned}\|u_m\|_{\mathscr{L}^{\alpha+2}_{\mathfrak{d},T}} &\lesssim \|u_0^m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{d}}}(\|\vartheta_m\|_{\mathcal{C}^{\alpha+2+\varepsilon}_{\mathfrak{n}}} + \|\xi_m\|_{\mathcal{C}^{\alpha+\varepsilon}_{\mathfrak{n}}} + 1)\\ &+ T^{(\alpha+2)/2}\|u_0^m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{d}}}(\|\vartheta_m\|_{\mathcal{C}^{\alpha+2+\varepsilon}_{\mathfrak{n}}} + \|\xi_m\|_{\mathcal{C}^{\alpha+\varepsilon}_{\mathfrak{n}}} + 1)(\|\vartheta_m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{n}}} + \|\xi_m\|_{\mathcal{C}^{\alpha}_{\mathfrak{n}}}\\ &+ \|\xi_m\|_{\mathcal{C}^{\alpha}_{\mathfrak{n}}}\|\vartheta_m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{n}}} + \|\vartheta_m\odot\xi_m - c_m\|_{\mathcal{C}^{2\alpha+2}_{\mathfrak{n}}}) + T^{\varepsilon/2}\|u_0^m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{d}}}(1+\|\vartheta_m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{n}}})\|\xi_m\|_{\mathcal{C}^{\alpha+4\varepsilon}_{\mathfrak{n}}}.\end{aligned}$$

The sufficiency of the time $T>0$ does not depend on u_0^m, just on the noise terms in appropriate norms.

By repeating Steps 1 and 2, we also get the following bounds:

$$\|u_m - Pu_0^m\|_{\mathscr{L}^{\alpha+2}_{\mathfrak{d},T}} \lesssim \|u^\sharp_m - Pu_0^m\|_{\mathscr{L}^{\alpha+2}_{\mathfrak{d},T}} + \|u_0^m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{d}}}(\|\xi_m\|_{\mathcal{C}^{\alpha+\varepsilon}_{\mathfrak{n}}} + \|\vartheta_m\|_{\mathcal{C}^{\alpha+2+\varepsilon}_{\mathfrak{n}}}),$$

and

$$\|u^\sharp_m - Pu_0^m\|_{\mathscr{L}^{\alpha+2}_{\mathfrak{d},T}} \lesssim \|u_0^m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{d}}}\|\vartheta_m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{n}}} + \|\Phi^\sharp_m\|_{M_T^{(\alpha+2)/2}\mathcal{C}^{2(\alpha+2)-2}_{\mathfrak{d}}}.$$

Those yield the following:

Lemma 1.47
With the notation of Lemma 1.45, we get the bound for T sufficiently small:

$$\begin{aligned}\|u_m - Pu_0^m\|_{\mathscr{L}^{\alpha+2}_{\mathfrak{d},T}} &\lesssim \|u_0^m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{d}}}(\|\vartheta_m\|_{\mathcal{C}^{\alpha+2+\varepsilon}_{\mathfrak{n}}} + \|\xi_m\|_{\mathcal{C}^{\alpha+\varepsilon}_{\mathfrak{n}}})\\ &+ T^{(\alpha+2)/2}\|u_0^m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{d}}}(\|\vartheta_m\|_{\mathcal{C}^{\alpha+2+\varepsilon}_{\mathfrak{n}}} + \|\xi_m\|_{\mathcal{C}^{\alpha+\varepsilon}_{\mathfrak{n}}} + 1)(\|\vartheta_m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{n}}} + \|\xi_m\|_{\mathcal{C}^{\alpha}_{\mathfrak{n}}}\\ &+ \|\xi_m\|_{\mathcal{C}^{\alpha}_{\mathfrak{n}}}\|\vartheta_m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{n}}} + \|\vartheta_m\odot\xi_m - c_m\|_{\mathcal{C}^{2\alpha+2}_{\mathfrak{n}}}) + T^{\varepsilon/2}\|u_0^m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{d}}}(1+\|\vartheta_m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{n}}})\|\xi_m\|_{\mathcal{C}^{\alpha+4\varepsilon}_{\mathfrak{n}}}.\end{aligned}$$

The sufficiency of the time $T>0$ does not depend on u_0^m, just on the noise terms in appropriate norms.

We can now close those bounds for arbitrary $T>0$.

Lemma 1.48
Let for $m \in \mathbb{N}$, $u_0^m \in I_{PAM}^m$. Then there exist some C_1^m, $C_2^m > 0$ such that for any $T > 0$,

$$\|u_m\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} \leq \|u_0^m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} C_1^m \exp(C_2^m T).$$

The constants C_1^m, C_2^m are uniformly bounded in m. What is more, there exists some constants $K(\boldsymbol{\theta}), A(\boldsymbol{\theta}) > 0$, for any $\boldsymbol{\theta} \in \mathfrak{X}_{\mathfrak{n}}^{\alpha+4\varepsilon}$, depending only on appropriate norms of $\boldsymbol{\theta}$, such that

$$\|u_m - Pu_0^m\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} \leq \|u_0^m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} K(\boldsymbol{\xi}_m) \exp(\log(K(\boldsymbol{\xi}_m))(A(\boldsymbol{\xi}_m)+1)T).$$

In particular, if $\boldsymbol{\theta}_k \to \boldsymbol{\theta} \in \mathfrak{X}_{\mathfrak{n}}^{\alpha+4\varepsilon}$ as $k \to \infty$, then $A(\boldsymbol{\theta}_k)$ and $K(\boldsymbol{\theta}_k)$ are bounded uniformly in k. Also, if $\boldsymbol{\theta}_k \to 0$ in $\mathfrak{X}_{\mathfrak{n}}^{\alpha+4\varepsilon}$, then $K(\boldsymbol{\theta}_k) \to 0$.

Proof
By Lemma 1.46 for $T \leq (A(\boldsymbol{\xi}_m)+1)^{-1}$,

$$\begin{aligned}
\|u_m\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} &\lesssim \|u_0^m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}(\|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2+\varepsilon}} + \|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+\varepsilon}} + 1) \\
&+ \|u_0^m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}(\|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2+\varepsilon}} + \|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+\varepsilon}} + 1)(\|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2}} + \|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha}} + \|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha}}\|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2}} \\
&+ \|\vartheta_m \odot \xi_m - c_m\|_{\mathcal{C}_{\mathfrak{n}}^{2\alpha+2}}) + \|u_0^m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}(1 + \|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2}})\|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+4\varepsilon}},
\end{aligned}$$

where $A(\boldsymbol{\xi}_m) > 0$ is some constant depending on the noise in appropriate norms. Note that we have bounded $T \leq 1$. Also, we may assume that the contribution in the suppressed constant by the absorptions carried out above can be bounded by e.g. 2 after increasing $A(\boldsymbol{\xi}_m)$. In short, for some $K_m > 1$ depending only on noise terms in appropriate norms, $\|u_m\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} \leq K_m \|u_0^m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}$.
Let $T > 0$ be arbitrary and let $N \in \mathbb{N}$ be large enough such that $T < N(A(\boldsymbol{\xi}_m)+1)^{-1}$. Then by iterating the bound,

$$\|u_m\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} \leq \|u_m\|_{\mathscr{L}_{\mathfrak{d},N(A(\boldsymbol{\xi}_m)+1)^{-1}}^{\alpha+2}} \leq K_m^N \|u_0^m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}.$$

We can choose $\lceil (A(\boldsymbol{\xi}_m)+1)T \rceil \leq (A(\boldsymbol{\xi}_m)+1)T + 1 =: N$. Consequently, using that $K_m = \exp(\log(K_m))$,

$$\|u_m\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} \leq \exp(\log(K_m)((A(\boldsymbol{\xi}_m)+1)T+1))\|u_0^m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}. \tag{8}$$

The second claim follows as the above by choosing

$$\begin{aligned}
K(\boldsymbol{\xi}_m) = C((&\|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+\varepsilon}} + \|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2+\varepsilon}}) + (\|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2+\varepsilon}} + \|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+\varepsilon}} + 1)(\|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2}} + \|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha}} \\
&+ \|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha}}\|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2}} + \|\vartheta_m \odot \xi_m - c_m\|_{\mathcal{C}_{\mathfrak{n}}^{2\alpha+2}}) + (1 + \|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2}})\|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+4\varepsilon}}),
\end{aligned}$$

for some constant $C > 0$.

□

Let us derive a bound on the $\mathscr{L}_{\mathfrak{d},T}^{2(\alpha+2)}$-norm of $u_m^\sharp$ for arbitrary $T > 0$ after some (small) amount of time has passed.

Lemma 1.49
Let $T > 0$ and $0 < \varepsilon < 1/4(-1-\alpha)$. Let θ_t be the shift operator, i.e. $(f\circ\theta_t)(s) := f(s+t)$. We define

$$k_1(\boldsymbol{\xi}_m) := C(1 + \|\xi_m\|_{\mathcal{C}_\mathfrak{n}^{\alpha+4\varepsilon}}),$$
$$k_2(\boldsymbol{\xi}_m) := C\left(\|\vartheta_m\|_{\mathcal{C}_\mathfrak{n}^{\alpha+2}} + \|\xi_m\|_{\mathcal{C}_\mathfrak{n}^{\alpha}} + \|\xi_m\|_{\mathcal{C}_\mathfrak{n}^{\alpha}}\|\vartheta_m\|_{\mathcal{C}_\mathfrak{n}^{\alpha+2}} + \|\vartheta_m \odot \xi_m - c_m\|_{\mathcal{C}_\mathfrak{n}^{2\alpha+2}}\right),$$

for some specific constant $C > 0$, see the proof below.
Let $0 < d(\boldsymbol{\xi}_m) < (1 + c\|\xi_m\|_{\mathcal{C}_\mathfrak{n}^{\alpha+4\varepsilon}})^{-1/\varepsilon} \wedge T$ be arbitrarily small, where c is another constant. Assume $N(T,\boldsymbol{\xi}_m) := \lfloor T/d(\boldsymbol{\xi}_m)\rfloor \neq T/d(\boldsymbol{\xi}_m)$. Then it holds that

$$\begin{aligned}
&\|u_m^\sharp \circ \theta_{T-N(T,\boldsymbol{\xi}_m)d(\boldsymbol{\xi}_m)}\|_{\mathscr{L}^{2(\alpha+2)-2\varepsilon}_{\mathfrak{d},N(T,\boldsymbol{\xi}_m)d(\boldsymbol{\xi}_m)}} \\
&\leq \sum_{j=0}^{N(T,\boldsymbol{\xi}_m)-1} \|u_m^\sharp(T - N(T,d(\boldsymbol{\xi}_m))d(\boldsymbol{\xi}_m))\|_{\mathcal{C}_\mathfrak{d}^{2(\alpha+2)-2\varepsilon}} k_1(\boldsymbol{\xi}_m)^{N(T,d(\boldsymbol{\xi}_m))-j} \\
&+ \|u_m\|_{\mathscr{L}^{\alpha+2}_{\mathfrak{d},T}} k_2(\boldsymbol{\xi}_m)\left(\sum_{i=0}^{N(T,d(\boldsymbol{\xi}_m))-j-1} k_1^i(\boldsymbol{\xi}_m)\right).
\end{aligned}$$

Proof
Assume $0 < \varepsilon < 1/4(-1-\alpha)$. Let $T > 0$ be arbitrary and let $0 < d(\boldsymbol{\xi}_m) < (1 + c\|\xi_m\|_{\mathcal{C}_\mathfrak{n}^{\alpha+4\varepsilon}})^{-1/\varepsilon} \wedge T$, where c is some constant as determined below. It follows by an elementary calculation that $u_m^\sharp \circ \theta_{T-d(\boldsymbol{\xi}_m)} = P(u_m^\sharp(T - d(\boldsymbol{\xi}_m))) + I(\Phi_m^\sharp \circ \theta_{T-d(\boldsymbol{\xi}_m)})$. Consequently, it holds that

$$\begin{aligned}
&\|u_m^\sharp \circ \theta_{T-d(\boldsymbol{\xi}_m)}\|_{\mathscr{L}^{2(\alpha+2)-2\varepsilon}_{\mathfrak{d},d(\boldsymbol{\xi}_m)}} \\
&\lesssim \|u_m^\sharp(T - d(\boldsymbol{\xi}_m))\|_{\mathcal{C}_\mathfrak{d}^{2(\alpha+2)-2\varepsilon}} + \|\Phi_m^\sharp \circ \theta_{T-d(\boldsymbol{\xi}_m)}\|_{C_{d(\boldsymbol{\xi}_m)}\mathcal{C}_\mathfrak{d}^{2(\alpha+2)-2-2\varepsilon}}.
\end{aligned}$$

In order to estimate terms involving the modified paraproduct, we simply bound e.g.

$$\begin{aligned}
&\|(u_m \otimes_T L\vartheta_m - L(u_m \otimes_T \vartheta_m)) \circ \theta_{T-d(\boldsymbol{\xi}_m)}\|_{C_{d(\boldsymbol{\xi}_m)}\mathcal{C}_\mathfrak{d}^{2(\alpha+2)-2-2\varepsilon}} \\
&\leq \|u_m \otimes_T L\vartheta_m - L(u_m \otimes_T \vartheta_m)\|_{C_T\mathcal{C}_\mathfrak{d}^{2(\alpha+2)-2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|\Phi_m^\sharp \circ \theta_{T-d(\boldsymbol{\xi}_m)}\|_{C_{d(\boldsymbol{\xi}_m)}\mathcal{C}_\mathfrak{d}^{2(\alpha+2)-2-2\varepsilon}} \\
&\lesssim \|u_m\|_{\mathscr{L}^{\alpha+2}_{\mathfrak{d},T}}\left(\|\vartheta_m\|_{\mathcal{C}_\mathfrak{n}^{\alpha+2}} + \|\xi_m\|_{\mathcal{C}_\mathfrak{n}^{\alpha}} + \|\xi_m\|_{\mathcal{C}_\mathfrak{n}^{\alpha}}\|\vartheta_m\|_{\mathcal{C}_\mathfrak{n}^{\alpha+2}} + \|\vartheta_m \odot \xi_m - c_m\|_{\mathcal{C}_\mathfrak{n}^{2\alpha+2}}\right) \\
&+ \|u_m^\sharp \circ \theta_{T-d(\boldsymbol{\xi}_m)}\|_{C_{d(\boldsymbol{\xi}_m)}\mathcal{C}_\mathfrak{d}^{2(\alpha+2)-4\varepsilon}}\|\xi_m\|_{\mathcal{C}_\mathfrak{n}^{\alpha+4\varepsilon}}.
\end{aligned}$$

Next,

$$\begin{aligned}&\|u_m^\sharp\circ\theta_{T-d(\boldsymbol{\xi}_m)}\|_{\mathscr{L}^{2(\alpha+2)-4\varepsilon}_{\mathfrak{d},d(\boldsymbol{\xi}_m)}}\\&\lesssim\|u_m^\sharp(T-d(\boldsymbol{\xi}_m))\|_{\mathcal{C}^{2(\alpha+2)-2\varepsilon}_{\mathfrak{d}}}+d(\boldsymbol{\xi}_m)^\varepsilon\|u_m^\sharp\circ\theta_{T-d(\boldsymbol{\xi}_m)}\|_{\mathscr{L}^{2(\alpha+2)-2\varepsilon}_{\mathfrak{d},d(\boldsymbol{\xi}_m)}}.\end{aligned}$$

Consequently by absorbing one term,

$$\begin{aligned}&\|\Phi_m^\sharp\circ\theta_{T-d(\boldsymbol{\xi}_m)}\|_{C_{d(\boldsymbol{\xi}_m)}\mathcal{C}_{\mathfrak{d}}^{2(\alpha+2)-2-2\varepsilon}}\\&\lesssim\|u_m\|_{\mathscr{L}^{\alpha+2}_{\mathfrak{d},T}}\left(\|\vartheta_m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{n}}}+\|\xi_m\|_{\mathcal{C}^{\alpha}_{\mathfrak{n}}}+\|\xi_m\|_{\mathcal{C}^{\alpha}_{\mathfrak{n}}}\|\vartheta_m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{n}}}+\|\vartheta_m\odot\xi_m-c_m\|_{\mathcal{C}^{2\alpha+2}_{\mathfrak{n}}}\right)\\&+\|u_m^\sharp(T-d(\boldsymbol{\xi}_m))\|_{\mathcal{C}^{2(\alpha+2)-2\varepsilon}_{\mathfrak{d}}}\|\xi_m\|_{\mathcal{C}^{\alpha+4\varepsilon}_{\mathfrak{n}}}.\end{aligned}$$

It follows that

$$\|u_m^\sharp\circ\theta_{T-d(\boldsymbol{\xi}_m)}\|_{\mathscr{L}^{2(\alpha+2)-2\varepsilon}_{\mathfrak{d},d(\boldsymbol{\xi}_m)}}\le\|u_m^\sharp(T-d(\boldsymbol{\xi}_m))\|_{\mathcal{C}^{2(\alpha+2)-2\varepsilon}_{\mathfrak{d}}}k_1(\boldsymbol{\xi}_m)+\|u_m\|_{\mathscr{L}^{\alpha+2}_{\mathfrak{d},T}}k_2(\boldsymbol{\xi}_m)$$

with

$$\begin{aligned}&k_1(\boldsymbol{\xi}_m):=C(1+\|\xi_m\|_{\mathcal{C}^{\alpha+4\varepsilon}_{\mathfrak{n}}}),\\&k_2(\boldsymbol{\xi}_m):=C\left(\|\vartheta_m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{n}}}+\|\xi_m\|_{\mathcal{C}^{\alpha}_{\mathfrak{n}}}+\|\xi_m\|_{\mathcal{C}^{\alpha}_{\mathfrak{n}}}\|\vartheta_m\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{n}}}+\|\vartheta_m\odot\xi_m-c_m\|_{\mathcal{C}^{2\alpha+2}_{\mathfrak{n}}}\right),\end{aligned}$$

for some constant $C>0$. We can now repeat the derivation above to get a bound for $\|u_m^\sharp\circ\theta_{T-id(\boldsymbol{\xi}_m)}\|_{\mathscr{L}^{2(\alpha+2)-2\varepsilon}_{\mathfrak{d},d(\boldsymbol{\xi}_m)}}$ with $i\in\mathbb{N}$ such that $id(\boldsymbol{\xi}_m)<T$. Assume $N(T,\boldsymbol{\xi}_m):=\lfloor T/d(\boldsymbol{\xi}_m)\rfloor\neq T/d(\boldsymbol{\xi}_m)$. Then, by using the bound $\|u_m^\sharp(T-d(\boldsymbol{\xi}_m))\|_{\mathcal{C}^{2(\alpha+2)-2\varepsilon}_{\mathfrak{d}}}\le\|u_m^\sharp\circ\theta_{T-2d(\boldsymbol{\xi}_m)}\|_{\mathscr{L}^{2(\alpha+2)-2\varepsilon}_{\mathfrak{d},d(\boldsymbol{\xi}_m)}}$, we get

$$\begin{aligned}\|u_m^\sharp\circ\theta_{T-d(\boldsymbol{\xi}_m)}\|_{\mathscr{L}^{2(\alpha+2)-2\varepsilon}_{\mathfrak{d},d(\boldsymbol{\xi}_m)}}&\le\|u_m^\sharp(T-N(T,d(\boldsymbol{\xi}_m))d(\boldsymbol{\xi}_m))\|_{\mathcal{C}^{2(\alpha+2)-2\varepsilon}_{\mathfrak{d}}}k_1(\boldsymbol{\xi}_m)^{N(T,\boldsymbol{\xi}_m)}\\&+\|u_m\|_{\mathscr{L}^{\alpha+2}_{\mathfrak{d},T}}k_2(\boldsymbol{\xi}_m)\left(\sum_{i=0}^{N(T,\boldsymbol{\xi}_m)-1}k_1^i(\boldsymbol{\xi}_m)\right).\end{aligned}$$

Finally, we can decompose for $T-N(T,\boldsymbol{\xi}_m)d(\boldsymbol{\xi}_m)\le t<T$,

$$u_m^\sharp(t)=\sum_{j=0}^{N(T,\boldsymbol{\xi}_m)-1}\mathbb{1}_{[T-(j+1)d(\boldsymbol{\xi}_m),T-jd(\boldsymbol{\xi}_m))}(t)u_m^\sharp(t).$$

The triangle inequality now yields the bound

$$\|u_m^\sharp\circ\theta_{T-N(T,\boldsymbol{\xi}_m)d(\boldsymbol{\xi}_m)}\|_{\mathscr{L}^{2(\alpha+2)-2\varepsilon}_{\mathfrak{d},N(T,\boldsymbol{\xi}_m)d(\boldsymbol{\xi}_m)}}\le\sum_{j=0}^{N(T,\boldsymbol{\xi}_m)-1}\|u_m^\sharp\circ\theta_{T-(j+1)d(\boldsymbol{\xi}_m)}\|_{\mathscr{L}^{2(\alpha+2)-2\varepsilon}_{\mathfrak{d},d(\boldsymbol{\xi}_m)}}.$$

To see this, let $\beta \in (0,1]$, $0 < t_1 < T$, and $0 \le s < t_1 < t \le T$. Then, $\|f(t) - f(s)\|_{L^\infty} \le \|f(t) - f(t_1)\|_{L^\infty} + \|f(t_1) - f(s)\|_{L^\infty}$, $|t-s|^\beta \ge |t-t_1|^\beta$ and $|t-s|^\beta \ge |t_1 - s|^\beta$. Consequently,

$$\begin{aligned}
&\|u_m^\sharp \circ \theta_{T-N(T,\boldsymbol{\xi}_m)d(\boldsymbol{\xi}_m)}\|_{\mathscr{L}^{2(\alpha+2)-2\varepsilon}_{\mathfrak{d},N(T,\boldsymbol{\xi}_m)d(\boldsymbol{\xi}_m)}} \\
&\le \sum_{j=0}^{N(T,\boldsymbol{\xi}_m)-1} \|u_m^\sharp(T - N(T,d(\boldsymbol{\xi}_m))d(\boldsymbol{\xi}_m))\|_{\mathcal{C}^{2(\alpha+2)-2\varepsilon}_{\mathfrak{d}}} k_1(\boldsymbol{\xi}_m)^{N(T,d(\boldsymbol{\xi}_m))-j} \\
&+ \|u_m\|_{\mathscr{L}^{\alpha+2}_{\mathfrak{d},T}} k_2(\boldsymbol{\xi}_m) \left(\sum_{i=0}^{N(T,d(\boldsymbol{\xi}_m))-j-1} k_1^i(\boldsymbol{\xi}_m) \right).
\end{aligned}$$

□

One can now combine this with the bound of Step 3 to get a closed expression.

1.6 Around the Space $I_{\mathbf{PAM}}$

We still assume that the initial conditions lie in the space $I^{\mathrm{m}}_{\mathrm{PAM}} = \mathrm{Dom}(\mathscr{H}_{\boldsymbol{\xi}_m}) \cap \mathcal{C}^{\alpha+2}_{\mathfrak{d}}$ in order to use the variation-of-constants representations together with Schauder estimates. The space $I^{\mathrm{m}}_{\mathrm{PAM}}$ however does not enjoy good properties, in particular it is not closed under taking squares. This is why, as a first step, we extend the above to other initial conditions.

Definition 1.50
We define I_{PAM} to be the closure of $I^m_{PAM} \subset \mathcal{C}^{\alpha+2}_{\mathfrak{d}}$, that is $I_{PAM} = \overline{I^m_{PAM}}^{\mathcal{C}^{\alpha+2}_{\mathfrak{d}}}$.

The space I_{PAM} is in turn to broad. However, we can identify a reasonably large and well-behaved subspace.

Definition 1.51
We define $\mathcal{C}^{(\alpha+2)+}_{\mathfrak{d}} := \bigcup_{0<\delta<1-(\alpha+2)} \mathcal{C}^{\alpha+2+\delta}_{\mathfrak{d}}$.

Lemma 1.52
It holds that $\mathcal{C}^{(\alpha+2)+}_{\mathfrak{d}} \subset I_{PAM}$.

Proof
Let $0 < \delta < 1-(\alpha+2)$ and $u \in \mathcal{C}^{\alpha+2+\delta}_{\mathfrak{d}}$. We need to show that we can approximate u with $u_k \in I^{\mathrm{m}}_{\mathrm{PAM}}$, $k \in \mathbb{N}$, in $\mathcal{C}^{\alpha+2}_{\mathfrak{d}}$. Define $u_k := P^{\mathrm{Dir}}_{1/k} u$. From what we have seen in the proofs of Lemma 1.36 and Lemma 1.44, $\widetilde{\Delta u_k} = \Delta P^{\mathrm{per}}_{1/k} \tilde{u} \in C(\mathbb{T}^d_{2L})$. Consequently, $u_k \in I^{\mathrm{m}}_{\mathrm{PAM}}$ and as we have seen in the proof of Theorem 1.17, $u_k \to u$ in $\mathcal{C}^{\alpha+2}_{\mathfrak{d}}$. This yields the claim.

□

We can now extend the bounds above to initial conditions in $\mathcal{C}^{(\alpha+2)+}_{\mathfrak{d}}$.

Lemma 1.53
The bounds of Lemma 1.46, Lemma 1.47, Lemma 1.48 and Lemma 1.49 hold true for $u_0^m \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$.

Proof
It suffices to show that the variation-of-constants representations of u_m and $u_m^\sharp$, and the relation $u_m = u_m \varolessthan_T \vartheta_m + u_m^\sharp$ carry over to $u_0^m \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$. Let $u_0^m \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$. By Lemma 1.52, we define for $k \in \mathbb{N}$, $u_0^{m,k} \in I_{\text{PAM}}^{\text{m}}$ such that $u_0^{m,k} \to u_0^m$ in $\mathcal{C}_{\mathfrak{d}}^{\alpha+2}$ as $k \to \infty$. We define u_m^k to be a solution to the mollified PAM started from $u_0^{m,k}$, with representation $u_m^k = Pu_0^{m,k} + I((\xi_m - c_m)u_m^k)$. As $u_0^{m,k} \to u_0^m$ in $\mathcal{C}_{\mathfrak{d}}^{\alpha+2}$, we get by [Lie89, Theorem 13.3], that $u_m^k \to u_m$ in $\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}$. Consequently by Lemma 1.39, $Pu_0^{m,k} \to Pu_0^m$ and $I((\xi_m - c_m)u_m^k) \to I((\xi_m - c_m)u_m)$ in $\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}$. Hence, $u_m = Pu_0^m + I((\xi_m - c_m)u_m)$. We define $u_m^{\sharp,k} := u_m^k - u_m^k \varolessthan_T \vartheta_m$. We have $u_m^{\sharp,k} = Pu_m^{\sharp,k}(0) + I\Phi_m^{\sharp,k}$ with $\Phi_m^{\sharp,k}$ being analogous to $\Phi_m^\sharp$. So, $u_m^{\sharp,k}(0) = u_0^{m,k} - u_0^{m,k} \varolessthan \vartheta_m \to u_0^m - u_0^m \varolessthan \vartheta_m$ in $\mathcal{C}_{\mathfrak{d}}^{\alpha+2}$ and by Lemma 1.44 it follows that $u_m^{\sharp,k} \to u_m^\sharp$ in $C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}$ with $u_m^\sharp := u_m - u_m \varolessthan_T \vartheta_m$.
Next, $Pu_m^{\sharp,k}(0) \to Pu_m^\sharp(0)$ in $\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}$. In order to show that $I\Phi_m^{\sharp,k} \to I\Phi_m^\sharp$ in $\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}$, we use the smoothness of the noise to establish first $\Phi_m^{\sharp,k} \to \Phi_m^\sharp$ in $C_T\mathcal{C}_{\mathfrak{d}}^\alpha$. Note that $\alpha < 2(\alpha+2) - 2$. Consequently, we can mostly mimic the derivation of (7) to get

$$\begin{aligned}\|\Phi_m^\sharp\|_{C_T\mathcal{C}_{\mathfrak{d}}^\alpha} \lesssim \|u_m\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}}(\|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2}} + \|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^\alpha} + \|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^\alpha}\|\vartheta_m\|_{\mathcal{C}_{\mathfrak{n}}^{\alpha+2}} + \|\xi_m \odot \vartheta_m - c_m\|_{\mathcal{C}_{\mathfrak{n}}^{2\alpha+2}}) \\ + \sup_{0\le t\le T}\|u_m^\sharp(t) \odot \xi_m\|_{\mathcal{C}_{\mathfrak{d}}^\alpha},\end{aligned}$$

with the non-natural bound $\|u_m^\sharp(t) \odot \xi_m\|_{\mathcal{C}_{\mathfrak{d}}^\alpha} \lesssim \|u_m^\sharp(t)\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}\|\xi_m\|_{\mathcal{C}_{\mathfrak{n}}^{2\alpha+2}}$. Using bilinearity and the results already established, we get $I\Phi_m^{\sharp,k} \to I\Phi_m^\sharp$ in $\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}$ by another application of Lemma 1.39.
Consequently, $u_m^\sharp = Pu_m^\sharp(0) + I\Phi_m^\sharp$. We have shown that u_m, $u_m^\sharp$ and $\Phi_m^\sharp$ admit exactly the same structure as before. Therefore, all results with initial conditions in $I_{\text{PAM}}^{\text{m}}$ carry over to solutions started from $\mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$.

□

1.7 The Dirichlet Parabolic Anderson Model

We can now derive the solution theory for the Parabolic Anderson Model with relative ease:
Let $T > 0$ be arbitrary. We first control the remainder $u_m^\sharp$ in $\mathscr{L}_{\mathfrak{d},T(\boldsymbol{\xi}_m)}^{(\alpha+2+\varepsilon)/2,2(\alpha+2)-2\varepsilon}$ until some sufficiently small time $0 < T(\boldsymbol{\xi}_m) < T$, only depending on the noise in appropriate norms, see Step 3 in the derivation of Lemma 1.46. By the above, $T(\boldsymbol{\xi}_m)$ is bounded below uniformly in m. From time $T(\boldsymbol{\xi}_m)$ onwards, we can control the remainder by Lemma 1.49 until the time horizon $T > 0$. The same also holds for $\Phi_m^\sharp$, see (7) and the proof of Lemma 1.49 for the appropriate bounds.
Let now $n, m \in \mathbb{N}$ and u_m, u_n be solutions respectively. By repeating the derivation above for $u_n - u_m$, using bilinearity, the bounds established in Lemma 1.53 and

Lemma 1.39, we can show the local Lipschitz continuity of the solution map $\mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+} \times \mathcal{C}_{\mathfrak{n}}^{\alpha+4\varepsilon} \times \mathcal{C}_{\mathfrak{n}}^{\alpha+2+4\varepsilon} \times \mathcal{C}_{\mathfrak{n}}^{2\alpha+2+8\varepsilon} \ni (u_0^m, \xi_m, \vartheta_m, \xi_m \odot \vartheta_m - c_m) \to (u_m, u_m^\sharp, \Phi_m^\sharp)$. Assume that $(u_0^m, \xi_m, \vartheta_m, \xi_m \odot \vartheta_m - c_m) \to (u_0, \xi, \vartheta, \Xi)$, $u_0 \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$. In particular, we may assume $(u_0^m, \xi_m, \vartheta_m, \xi_m \odot \vartheta_m - c_m)_{m\in\mathbb{N}}$ to be bounded. Hence, the solution map is Lipschitz continuous on a relevant subspace. We define $u \in C\mathcal{C}_{\mathfrak{d}}^{\alpha+2} \cap C_{\mathrm{loc}}^{(\alpha+2)/2} L^\infty$ to be the limit of the continuous extension of the solution map. We get:

Theorem 1.54
Let $\alpha \in (-4/3, -1)$, $\varepsilon \in (0, 1/4(-1-\alpha))$ and $u_0^m \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$, $m \in \mathbb{N}$, $u_0 \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$. Assume $(\xi, \Xi) \in \mathfrak{X}_{\mathfrak{n}}^{\alpha+4\varepsilon}$ and suppose that $(u_0^m, \xi_m, \sigma(D)\xi_m, \xi_m \odot \sigma(D)\xi_m - c_m,) \to (u_0, \xi, \vartheta, \Xi)$ with $\xi_m \in S_{\mathfrak{n}}([0,L]^2)$ smooth, $\vartheta_m := \sigma(D)\xi_m$, c_m as in the definition of $\mathfrak{X}_{\mathfrak{n}}^{\alpha+4\varepsilon}$. Let for any $m \in \mathbb{N}$, u_m be the unique solution to

$$\begin{cases} Lu_m = (\xi_m - c_m)u_m \quad \textit{in } (0,\infty) \times (0,L)^2, \\ u_m(0) = u_0^m \ \textit{in } [0,L]^2, \ u_m = 0 \ \textit{on } [0,\infty) \times \partial[0,L]^2. \end{cases}$$

Then there exists some $u \in C\mathcal{C}_{\mathfrak{d}}^{\alpha+2} \cap C_{loc}^{(\alpha+2)/2} L^\infty$ such that $u_m \to u$ in $\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}$ for any $T > 0$. What is more for any $T > 0$ there exists some function $u^\sharp$ such that $u_m^\sharp \to u^\sharp$ in $\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}$ as $m \to \infty$. Furthermore, there exists some $0 < T(\boldsymbol{\xi}) < T$, such that for arbitrarily small $0 < T' \leq T(\boldsymbol{\xi}) < T$, $u_m^\sharp \to u^\sharp$ in $\mathscr{L}_{\mathfrak{d},T'}^{(\alpha+2+\varepsilon)/2, 2(\alpha+2)-2\varepsilon}$ and $u_m^\sharp \circ \theta_{T'} \to u^\sharp \circ \theta_{T'}$ in $\mathscr{L}_{\mathfrak{d},T-T'}^{2(\alpha+2)-2\varepsilon}$.
The solution u only depends on $(u_0, \xi, \vartheta, \Xi)$ but not on the approximating family. We call it a paracontrolled solution to the Parabolic Anderson Model

$$\begin{cases} \partial_t u = \mathscr{H}_{\boldsymbol{\xi}} u \ \textit{in } (0,\infty) \times (0,L)^2, \\ u(0) = u_0 \ \textit{in } [0,L]^2, \ u = 0 \ \textit{on } [0,\infty) \times \partial[0,L]^2. \end{cases}$$

We will often use the notation $u_m(t) = T_t^m u_0^m$ and $u(t) = T_t u_0$. Note that T applied to functions always refers to the solution operator, not the time horizon.

As in Section 1.6 for $\delta > 0$,

$$\mathcal{C}_{\mathfrak{d}}^{\alpha+2+\delta} \subset \overline{\mathcal{C}_{\mathfrak{d}}^{\alpha+2+\delta/2} \cap \mathrm{Dom}(\mathscr{H}_{\boldsymbol{\xi}_m})}^{\mathcal{C}_{\mathfrak{d}}^{\alpha+2+\delta/2}}.$$

Also, the regularity of white noise is not sharp. Hence if we assume that $u_0 \in \mathcal{C}_{\mathfrak{d}}^{\alpha+2+\delta}$, $0 < \delta < 1 - (\alpha+2)$, then we can show that as above, $T_s u_0 \in \mathcal{C}_{\mathfrak{d}}^{\alpha+2+\delta/2}$, $s > 0$. Consequently, expressions such as $T_t T_s u_0$ and even $T_t (T_s u_0)^2$ are well-defined.
The solution also retains its paracontrolled structure:

Lemma 1.55
Let u be the solution constructed in Theorem 1.54. It holds that

$$u = Pu_0 + I(\boldsymbol{\xi} \diamond u), \quad u = u \otimes_T \vartheta + u^\sharp, \quad u^\sharp = Pu_0^\sharp + I\Phi^\sharp, \quad u_0^\sharp = u_0 - u_0 \otimes \vartheta,$$

where

$$\boldsymbol{\xi} \diamond u := \xi \otimes u + u \otimes \xi + \xi \odot u^\sharp + (u \otimes_T \vartheta - u \otimes \vartheta) \odot \xi + u\Xi + \mathscr{R}(u, \xi, \vartheta),$$

and

$$\begin{aligned}\Phi^\sharp &:= (u \otimes_T L\vartheta - L(u \otimes_T \vartheta)) + (u \otimes \xi - u \otimes L\vartheta) + (u \otimes L\vartheta - u \otimes_T L\vartheta) + \xi \otimes u \\ &\quad + \mathscr{R}(u, \xi, \vartheta) + u\Xi + u^\sharp \odot \xi + (u \otimes_T \vartheta - u \otimes \vartheta) \odot \xi.\end{aligned}$$

Proof
Let $0 < T(\boldsymbol{\xi}) < T$. It follows by the above that $u_m \to u$ in $\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}$, $u_m^\sharp \to u^\sharp$ in $\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}$, in $\mathscr{L}_{\mathfrak{d},T(\boldsymbol{\xi})}^{(\alpha+2+\varepsilon)/2,2(\alpha+2)-2\varepsilon}$, and from $T(\boldsymbol{\xi})$ onwards in $\mathscr{L}_{\mathfrak{d},T-T(\boldsymbol{\xi})}^{2(\alpha+2)-2\varepsilon}$.
As $u_m = u_m \otimes_T \vartheta_m + u_m^\sharp$, it follows by Lemma 1.44, that $u = u \otimes_T \vartheta + u^\sharp$. Next by Theorem 1.16, Theorem 1.17, Theorem 1.22 and Lemma 1.44, $(\xi_m - c_m)u_m \to \boldsymbol{\xi} \diamond u$ in $M_{T(\boldsymbol{\xi})}^{(\alpha+2+\varepsilon)/2}\mathcal{C}_{\mathfrak{d}}^{\alpha}$, and from $T(\boldsymbol{\xi})$ onwards in $C_{T-T(\boldsymbol{\xi})}\mathcal{C}_{\mathfrak{d}}^{\alpha}$. Consequently, we can combine both to get convergence in $M_T^{(\alpha+2+\varepsilon)/2}\mathcal{C}_{\mathfrak{d}}^{\alpha}$. Therefore, by Lemma 1.40, $u = Pu_0 + I(\boldsymbol{\xi} \diamond u)$. By the results cited above, $\Phi_m^\sharp \to \Phi^\sharp$ in $M_T^{(\alpha+2+\varepsilon)/2}\mathcal{C}_{\mathfrak{d}}^{2(\alpha+2)-2\varepsilon-2}$. It follows by Lemma 1.40, $I\Phi_m^\sharp \to I\Phi^\sharp$ in $\mathscr{L}_{\mathfrak{d},T}^{(\alpha+2+\varepsilon)/2,2(\alpha+2)-2\varepsilon}$. Also, $u_m^\sharp(0) = u_0^m - u_0^m \otimes \vartheta_m \to u_0 - u_0 \otimes \vartheta$ in $\mathcal{C}_{\mathfrak{d}}^{\alpha+2}$. Again by Lemma 1.40, $Pu_m^\sharp(0) \to Pu_0^\sharp$ in $\mathscr{L}_{\mathfrak{d},T}^{(\alpha+2+\varepsilon)/2,2(\alpha+2)-2\varepsilon}$. Hence, $u^\sharp = Pu_0^\sharp + I\Phi^\sharp$, which yields the claim.

□

The following results give some basic properties of the PAM. We first establish the maximum principle.

Lemma 1.56
Let $u_0 \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$, $u_0 \geq 0$. Then it follows that $T_t u_0 \geq 0$ for any $t \geq 0$.

Proof
Let $u_0 \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$, $u_0 \geq 0$. By applying the maximum principle [Eva10, Theorem 7.1.4.9] to the mollified PAM started from u_0, the claim follows by the convergence of Theorem 1.54.

□

The following is an analogue of Lemma 1.48 for our paracontrolled solution.

Lemma 1.57
Let $u_0 \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$. Then there exists some $C_1, C_2 > 0$ such that for any $T > 0$,

$$\|u\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} \leq C_1 \|u_0\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} \exp(C_2 T).$$

Also, with the same A, K as in Lemma 1.48,

$$\|u - Pu_0\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} \leq \|u_0\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} K(\boldsymbol{\xi}) \exp(\log(K(\boldsymbol{\xi}))(A(\boldsymbol{\xi}) + 1)T).$$

Proof
The paracontrolled solution u has by Lemma 1.55 the same structure as the u_m. The bounds follow exactly as in Lemma 1.48.

□

The continuous dependence of the solution on the initial condition holds:

Lemma 1.58
Let $T > 0$. Then it holds that $\mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+} \ni u_0 \mapsto Tu_0 \in \mathscr{L}_{\mathfrak{d},T}^{\alpha+2}$ is continuous.

Proof
Let $u_0 \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$, $u_k \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$, $k \in \mathbb{N}$, such that $u_k \to u_0 \in \mathcal{C}_{\mathfrak{d}}^{\alpha+2}$. We get

$$\begin{aligned}\|Tu_0 - Tu_k\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} \leq \|Tu_0 - T^m u_0\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} + \|T^m u_0 - T^m u_k\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} \\ + \|T^m u_k - Tu_k\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}}.\end{aligned} \tag{9}$$

Let $\varepsilon > 0$. Fix $k \in \mathbb{N}$ large enough such that $\|u_0 - u_k\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} < \varepsilon$. The convergence of the second term in (9) follows by Lemma 1.48 and bilinearity, since for some $C_1^m, C_2^m > 0$ that are bounded in m,

$$\|T^m u_0 - T^m u_k\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} \leq \|u_0 - u_k\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} C_1^m \exp(C_2^m T).$$

Upon choosing m larger if needed, we can bound the first and the third term in (9) for k fixed by ε, using bilinearity and Lemma 1.57.

□

Next we need to define a space on which $\mathscr{H}_{\boldsymbol{\xi}}$ enjoys much better regularity properties than previously discussed. It follows that for $t > 0$ and $u_0^m \in I_{\text{PAM}}^{\text{m}}$,

$$\mathscr{H}_{\boldsymbol{\xi}_m} \int_0^t T_s^m u_0^m ds = T_t^m u_0^m - u_0^m.$$

Therefore we define with some slight abuse of notation for $u_0 \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$,

$$\mathscr{H}_{\boldsymbol{\xi}} \int_0^t T_s u_0 ds := T_t u_0 - u_0. \tag{10}$$

With this it follows as in Theorem 1.54 that if $u_0^m \to u_0$ in $\mathcal{C}_{\mathfrak{d}}^{\alpha+2}$ as $m \to \infty$, then

$$\int_0^t T_s^m u_0^m ds \to \int_0^t T_s u_0 ds, \quad \mathscr{H}_{\boldsymbol{\xi}_m} \int_0^t T_s^m u_0^m ds \to \mathscr{H}_{\boldsymbol{\xi}} \int_0^t T_s u_0 ds \quad \text{in} \quad \mathcal{C}_{\mathfrak{d}}^{\alpha+2}. \tag{11}$$

Definition 1.59
Let for $t > 0$, $u_0^m \in I_{PAM}^m$, $u_0 \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$, $A_t^m u_0^m := \int_0^t T_s^m u_0^m ds$ and $A_t u_0 := \int_0^t T_s u_0 ds$.

We define the spaces

$$D_{\mathscr{H}_{\xi_m}} := \{A_t^m u_0^m | u_0^m \in I_{PAM}^m, u_0^m \geq 0, u_0^m \neq 0, t > 0\},$$
$$D_{\mathscr{H}_\xi} := \{A_t u_0 | u_0 \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}, u_0 \geq 0, u_0 \neq 0, t > 0\}.$$

We define $\mathscr{H}_\xi A_t u_0 := T_t u_0 - u_0$ *and for* $t > s$, $\mathscr{H}_\xi(A_t u_0 - A_s u_0) := T_t u_0 - T_s u_0$. *For arbitrary* $t, s > 0$, *we define* $\mathscr{H}_\xi(A_t u_0 + A_s u_0) := \mathscr{H}_\xi A_t u_0 + \mathscr{H}_\xi A_s u_0$.

The space $D_{\mathscr{H}_\xi}$ is dense in the non-negative, non-zero functions of $\mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$ by the following argument:

Lemma 1.60 [PR19, Lemma 3.5]
Let $u_0 \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$, $u_0 \geq 0$, $u_0 \neq 0$. *We define for* $k \in \mathbb{N}$, $u_k \in D_{\mathscr{H}_\xi}$ *by* $u_k := A_{1/k}(k u_0)$. *Then it holds that* $\|u_0 - u_k\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} \to 0$.

Proof
The claim follows by the continuity of $r \mapsto T_r u_0$ in $\mathcal{C}_{\mathfrak{d}}^{\alpha+2}$, since $u_k = k \int_0^{1/k} T_r u_0 dr$.

□

1.8 The Backwards Dirichlet Parabolic Anderson Model with Forcing

In this section we consider the equation given by

$$\begin{cases} (\partial_s + \mathscr{H}_\xi) u^t = f \text{ in } (0,t) \times (0,L)^2, \\ u^t(t) = u_0 \text{ in } [0,L]^2, \ u^t = 0 \text{ on } [0,t] \times \partial[0,L]^2, \end{cases}$$

where $t > 0$, $u_0 \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$, $u_0 \geq 0$, $u_0 \neq 0$, and $f \in C([0,t], \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+})$, $f \geq 0$, $f(s) \neq 0$, $0 \leq s \leq t$.
This equation will be useful in an intermediate step towards the uniqueness of the killed rough SBM. The following arguments are essentially due to [PR19, Lemma 4.6], with modifications to treat the Dirichlet boundary conditions.
Assume first that $u_0 \in D_{\mathscr{H}_\xi}$ and $f : [0,t) \to D_{\mathscr{H}_\xi}$ is *piecewise constant*. We then define a solution in the mild formulation by

$$u^t(s) := T_{t-s} u_0 - \int_s^t T_{r-s} f(r) dr.$$

Assume we have $u_0 = A_l v$, $v \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$, $v \geq 0$, $v \neq 0$, $l > 0$. Note that $T_{t-s} u_0 = \int_{t-s}^{t-s+l} T_r v dr$. Using that f is piecewise constant, we get for the redefined $\mathscr{H}_\xi$,

$$\mathscr{H}_\xi u^t(s) = T_{t-s+l} v - T_{t-s} v - \int_s^t \mathscr{H}_\xi T_{r-s} f(r) dr$$

and

$$\partial_s u^t(s) = -T_{t-s+l}v + T_{t-s}v + f(s) + \int_s^t \mathscr{H}_{\boldsymbol{\xi}} T_{r-s} f(r)dr.$$

Consequently, u^t is a pointwise solution for the redefined $\mathscr{H}_{\boldsymbol{\xi}}$.
Let $u_0 \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$, $u_0 \geq 0$, $u_0 \neq 0$, and $f \in C([0,t], \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+})$, $f \geq 0$, $f(s) \neq 0$, $0 \leq s \leq t$. By Lemma 1.60, there exist some $u_0^k \in D_{\mathscr{H}_{\boldsymbol{\xi}}}$, $f^k : [0,t) \to D_{\mathscr{H}_{\boldsymbol{\xi}}}$ piecewise constant, $k \in \mathbb{N}$, such that $u_0^k \to u_0$ in $\mathcal{C}_{\mathfrak{d}}^{\alpha+2}$ and $f^k \to f$ pointwise a.e. and uniformly bounded in time. In particular, u_0^k, f^k can be realised as

$$u_0^k := k \int_0^{1/k} T_r u_0 dr, \quad f^k(s) = \sum_{i=0}^{k-1} \mathbb{1}_{[t_i^k, t_{i+1}^k)}(s) k \int_0^{1/k} T_z f(t_i^k) dz,$$

for some partition $(t_i^k)_{i \in \{1,\dots,k\}}$, $t_0^k = 0$, $t_k^k = t$, such that $\lim_{k\to\infty} \sup_{i\in\{1,\dots,k\}} t_i^k - t_{i-1}^k = 0$.
To simplify the notation, we denote for any $s \in [0,t)$, $\lfloor s \rfloor_k = \max\{t_i^k | t_i^k \leq s\}$.
By the uniform continuity of $[0,t] \ni s \mapsto T_{t-s}u_0 \in \mathcal{C}_{\mathfrak{d}}^{\alpha+2}$, it follows that in $C([0,t], \mathcal{C}_{\mathfrak{d}}^{\alpha+2})$,

$$\left[s \mapsto T_{t-s}u_0^k = k \int_{t-s}^{t-s+1/k} T_r u_0 dr \right] \to [s \mapsto T_{t-s}u_0].$$

Next we claim that in $C([0,t], \mathcal{C}_{\mathfrak{d}}^{\alpha+2})$,

$$\left[s \mapsto \int_s^t T_{r-s} f^k(r) dr \right] \to \left[s \mapsto \int_s^t T_{r-s} f(r) dr \right].$$

Proof of the claim: We apply Fubini's theorem to get

$$\begin{aligned} &\int_s^t T_{r-s} f^k(r) dr = \int_s^t k \int_{r-s}^{r-s+1/k} T_z f(\lfloor r \rfloor_k) dz dr \\ &= \int_{1/k}^{t-s} \int_{z+s-1/k}^{z+s} k T_z f(\lfloor r \rfloor_k) dr dz = \int_{1/k}^{t-s} T_z \left(\int_{z+s-1/k}^{z+s} k f(\lfloor r \rfloor_k) dr \right) dz. \end{aligned}$$

We have

$$\begin{aligned} &\int_{1/k}^{t-s} T_z \left(\int_{z+s-1/k}^{z+s} k f(\lfloor r \rfloor_k) dr \right) dz \\ &= \int_{1/k}^{t-s} T_z f(z+s) dz + \int_{1/k}^{t-s} T_z \left(\int_{z+s-1/k}^{z+s} k f(\lfloor r \rfloor_k) dr - f(z+s) \right) dz. \end{aligned}$$

For the second term we choose by the uniform continuity of f for $\varepsilon > 0$, k sufficiently large uniformly in s, such that

$$\left\| \int_{z+s-1/k}^{z+s} k f(\lfloor r \rfloor_k) dr - f(z+s) \right\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} < \varepsilon.$$

Therefore for some $C > 0$,

$$\left\| \int_{1/k}^{t-s} T_z \left(\int_{z+s-1/k}^{z+s} k f(\lfloor r \rfloor_k) dr - f(z+s) \right) dz \right\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} \lesssim \int_{1/k}^{t-s} \exp(Cz) \varepsilon dz.$$

For the first term,

$$\begin{aligned} \left\| \int_{1/k}^{t-s} T_z f(z+s) dz - \int_0^{t-s} T_z f(z+s) dz \right\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} &\leq \int_s^{s+1/k} \| T_{z'-s} f(z') \|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} dz' \\ &\lesssim \int_s^{s+1/k} \| f \|_{C_t \mathcal{C}_{\mathfrak{d}}^{\alpha+2}} \exp(C(z'-s)) dz'. \end{aligned}$$

This proves the claim. We define

$$u_k^t(s) := T_{t-s} u_0^k - \int_s^t T_{r-s} f^k(r) dr.$$

Then by the above, $u_k^t \to u^t$ in $C([0,t], \mathcal{C}_{\mathfrak{d}}^{\alpha+2})$, where

$$u^t(s) := T_{t-s} u_0 - \int_s^t T_{r-s} f(r) dr.$$

We call this u^t a solution to the backwards Dirichlet Parabolic Anderson Model with forcing.

1.9 The Evolution Equation for the Killed Mollified Super Brownian Motion

In this section we construct and analyse the solution of the so-called 'Evolution Equation for the killed mollified Super Brownian Motion':

$$\begin{cases} \partial_t \phi_m(t) = \mathscr{H}_{\xi_m} \phi_m(t) - \frac{1}{2} \phi_m(t)^2 \text{ in } (0,T) \times (0,L)^2, \\ \phi_m(0) = \gamma \psi \text{ in } [0,L]^2, \phi_m(t) = 0 \text{ on } [0,T] \times \partial [0,L]^2, \end{cases}$$

for $\psi \in I_{\text{PAM}}^{\text{m}}$, $\psi \geq 0$, $t > 0$ and $\gamma > 0$ sufficiently small, i.e.

$$U_t^m(\gamma\psi) := \phi_m(t) = T_t^m \gamma \psi - \frac{1}{2} \int_0^t T_s^m ((U_{t-s}^m(\gamma\psi))^2) ds. \tag{12}$$

Our approach is based on Wild sums, as presented in [Eth00, Chapter 2.1]. The advantage of this representation is that the influence of the initial condition will be explicit, a fact which we will use to identify the moments of the killed mollified Super Brownian Motion. We define $S^m : C_T \mathcal{C}_{\mathfrak{d}}^{\alpha+2} \times C_T \mathcal{C}_{\mathfrak{d}}^{\alpha+2} \to C_T \mathcal{C}_{\mathfrak{d}}^{\alpha+2}$ by

$$S^m(f,g)(t) := -\frac{1}{2} \int_0^t T_s^m (f(t-s,\cdot) g(t-s,\cdot)) ds.$$

If we set up a Picard iteration we get,

$$
\begin{aligned}
\phi_0^m(t) :&= \gamma T_t^m \psi, \\
\phi_1^m(t) :&= \phi_0^m(t) + S^m(\phi_0^m, \phi_0^m)(t), \\
\phi_2^m(t) :&= \phi_0^m(t) + S^m(\phi_1^m, \phi_1^m)(t) \\
&= \phi_0^m(t) + S^m((\phi_0^m + S^m(\phi_0^m, \phi_0^m)), (\phi_0^m + S^m(\phi_0^m, \phi_0^m)))(t) \\
&= \phi_0^m(t) + S^m(\phi_0^m, \phi_0^m)(t) + S^m(\phi_0^m, S^m(\phi_0^m, \phi_0^m))(t) + S^m(S(\phi_0^m, \phi_0^m), \phi_0^m)(t) \\
&\quad + S^m(S^m(\phi_0^m, \phi_0^m), S^m(\phi_0^m, \phi_0^m))(t), \\
&\dots .
\end{aligned}
$$

We denote

$$
\begin{aligned}
&\gamma^{-1}\phi_0^m = \bullet, \\
&\gamma^{-2} S^m(\phi_0^m, \phi_0^m) = \curlyvee, \\
&\gamma^{-3} S^m(\phi_0^m, S^m(\phi_0^m, \phi_0^m)) = {}^{\bullet}\curlyvee^{\curlyvee}, \\
&\gamma^{-3} S^m(S^m(\phi_0^m, \phi_0^m), \phi_0^m) = {}^{\curlyvee}\curlyvee^{\bullet}, \\
&\gamma^{-4} S^m(S^m(\phi_0^m, \phi_0^m), S(\phi_0^m, \phi_0^m)) = {}^{\curlyvee}\curlyvee^{\curlyvee}, \\
&\dots .
\end{aligned}
$$

We denote the order of the tree τ, $|\tau|$, to be the number of leaves of τ, i.e. the number of vertices with at most one neighbour. On the other hand, when we write $|\tau(t,x)|$ then we mean the absolute value of the number $\tau(t,x) \in \mathbb{R}$. In general, $S^m(\tau_1, \tau_2)$ can be constructed by joining the roots of τ_1, τ_2:

$$S^m(\tau_1, \tau_2) =: {}^{\tau_1}\curlyvee^{\tau_2}.$$

We define $\mathcal{T}$ to be the set of ordered binary rooted trees, which can be constructed recursively by the following rule: Let $\mathcal{T}^0 := \{\bullet\}$. We then define for $n \in \mathbb{N}$, $\mathcal{T}^n := \{{}^{\tau_1}\curlyvee^{\tau_2} | \tau_1, \tau_2 \in \mathcal{T}^{n-1}\}$ and $\mathcal{T} = \cup_{n \in \mathbb{N}} \mathcal{T}^n$. Note that we do not identify isomorphic trees for combinatorical reasons. We can then expand the solution to (12) formally in γ as a power series in terms of such binary trees:

$$\phi_m(t,x) = \sum_{\tau \in \mathcal{T}} \gamma^{|\tau|} \tau,$$

where we identify $\bullet = \gamma^{-1}\phi_0^m$ and for τ such that $|\tau| \geq 2$, $\tau = {}^{\tau_1}\curlyvee^{\tau_2} = S^m(\tau_1, \tau_2)$. The following two lemmas show that this expression is not just formal:

Lemma 1.61 [Eth00, Lemma A.4]
Let for $n \in \mathbb{N}$, $C(n)$ be the number of ordered binary rooted trees of order n. Then it holds that the formal power series $\sum_{n \in \mathbb{N}} C(n)\gamma^n$ has radius of convergence $1/4$.

Proof
By considering subtrees τ_1, τ_2 of τ and the decomposition $\tau = {}^{\tau_1}\!\vee^{\tau_2}$, we get the relation

$$C(n) = \sum_{j=1}^{n-1} C(j)C(n-j), \quad C(1) = 1.$$

Note that the series $C_n := C(n+1)$ for $n \in \mathbb{N} \cup \{0\}$ are the Catalan numbers. Let $f : \mathbb{R} \to \mathbb{R}$ be the formal power series given by $f(\gamma) = \sum_{n=1}^{\infty} C(n)\gamma^n$, $f(0) = 0$. The formal generating function for the Catalan numbers, $F(\gamma) = \sum_{n=0}^{\infty} C_n \gamma^n$, has radius of convergence 1/4 and is explicitly given by $F(\gamma) = 2(1 + \sqrt{1-4\gamma})^{-1}$. It holds that $f(\gamma) = \gamma F(\gamma)$ and therefore this function has radius of convergence 1/4 as well and is given by $f(\gamma) = 1/2(1 - \sqrt{1-4\gamma})$, using the relation $(1+\sqrt{1-4\gamma})(1-\sqrt{1-4\gamma}) = 4\gamma$.
□

Lemma 1.62 [Eth00, Lemma 2.7]
Let for $n, m \in \mathbb{N}$, $t > 0$, $x \in [0, L]^2$, $a_n^m(t,x) := \sum_{\tau \in \mathcal{T}, |\tau| = n} \tau(t,x)$. Assume that for $t > 0$, $\sup_{0 \le s \le t} \|a_1^m(s,\cdot)\|_{L^\infty} < \infty$. Then it holds that the power series

$$\gamma \mapsto \sum_{n \in \mathbb{N}} a_n^m(t,x)\gamma^n$$

has a non-trivial radius of convergence for fixed t, uniformly in x.

Proof
This proof is an adaptation of [Eth00, Lemma 2.7] to our setting, building on some more careful estimates.
We have seen in Lemma 1.61 that the power series $\gamma \mapsto \sum_{n=1}^{\infty} C(n)\gamma^n$ has radius of convergence 1/4. Therefore it suffices to show that there exists for any $t > 0$ some $K_m(t) > 0$ such that for any tree $\tau \in \mathcal{T}$ and $x \in [0, L]^2$, $|\tau(t,x)| = (-1)^{|\tau|+1}\tau(t,x) \le \max\{t, 1\}^{|\tau|-1} K_m(t)^{|\tau|}$. Let

$$K_m(t) := \max\left\{ \sup_{0 \le s \le t} \|a_1^m(s,\cdot)\|_{L^\infty}, 1 \right\}.$$

We use induction over $|\tau| = n \in \mathbb{N}$. For $n = 1$, $\bullet(t,x) = a_1^m(t,x) \le K_m(t)$. Let now $\tau \in \mathcal{T}$ be such that $|\tau| = n+1$, $n \in \mathbb{N}$, and $\tau = {}^{\tau_1}\!\vee^{\tau_2}$ with $\tau_1, \tau_2 \in \mathcal{T}$. We assume first that $|\tau_1| = 1$ and $|\tau_2| = n$. Then if $t > 0$, using the monotonicity of $s \mapsto K_m(s)$,

$$\begin{aligned}
0 &\le (-1)^{|\tau|+1} {}^{\tau_1}\!\vee^{\tau_2}(t,x) = \frac{1}{2}\int_0^t T_{t-s}^m((-1)^{|\tau_1|+1} T_s^m \psi (-1)^{|\tau_2|+1}\tau_2(s))(x) ds \\
&\le K_m^{|\tau_2|}(t) \int_0^t T_t^m \psi(x) \max\{s,1\}^{|\tau_2|-1} ds \le K_m^{|\tau_2|}(t) \max\{t,1\}^{|\tau_2|-1} t K_m(t) \\
&\le \max\{t,1\}^{|\tau|-1} K_m^{|\tau|}(t).
\end{aligned}$$

Assume now that both $|\tau_1|, |\tau_2| \geq 2$ and $\tau_1 = {}^{\tau_3}\!\vee{}^{\tau_4}$ with $\tau_3, \tau_4 \in \mathcal{T}$. Then

$$\begin{aligned}
&0 \leq (-1)^{|\tau|+1}\, {}^{\tau_1}\!\vee{}^{\tau_2}(t,x) = \frac{1}{2}\int_0^t T^m_{t-s}((-1)^{|\tau_1|+1}\tau_1(s)(-1)^{|\tau_2|+1}\tau_2(s))(x)ds \\
&\leq K_m^{|\tau_2|}(t)\max\{t,1\}^{|\tau_2|-1}\int_0^t T^m_{t-s}((-1)^{|\tau_1|+1}\tau_1(s))(x)ds \\
&= K_m^{|\tau_2|}(t)\max\{t,1\}^{|\tau_2|-1}\int_0^t T^m_{t-s}\frac{1}{2}\int_0^s T^m_r((-1)^{|\tau_3|+1}\tau_3(s-r)(-1)^{|\tau_4|+1}\tau_4(s-r))(x)drds \\
&= K_m^{|\tau_2|}(t)\max\{t,1\}^{|\tau_2|-1}\frac{1}{2}\int_0^t\int_0^s T^m_{r+t-s}((-1)^{|\tau_3|+1}\tau_3(s-r)(-1)^{|\tau_4|+1}\tau_4(s-r))(x)drds.
\end{aligned}$$

We substitute $z = r+t-s$ and get by Fubini's theorem,

$$\begin{aligned}
&\frac{1}{2}\int_0^t\int_0^s T^m_{r+t-s}((-1)^{|\tau_3|+1}\tau_3(s-r)(-1)^{|\tau_4|+1}\tau_4(s-r))(x)drds \\
&= \frac{1}{2}\int_0^t\int_{t-s}^t T^m_z((-1)^{|\tau_3|+1}\tau_3(t-z)(-1)^{|\tau_4|+1}\tau_4(t-z))(x)dzds \\
&= \frac{1}{2}\int_0^t zT^m_z((-1)^{|\tau_3|+1}\tau_3(t-z)(-1)^{|\tau_4|+1}\tau_4(t-z))(x)dz \\
&\leq \max\{t,1\}(-1)^{|\tau_1|+1}\tau_1(t,x) \leq \max\{t,1\}^{|\tau_1|}K_m^{|\tau_1|}(t).
\end{aligned}$$

Therefore, $0 \leq (-1)^{|\tau|+1}\, {}^{\tau_1}\!\vee{}^{\tau_2}(t,x) \leq \max\{t,1\}^{|\tau|-1}K_m^{|\tau|}(t)$. Together with the above this yields the claim.

□

Explicitly we have $a_1^m(t,x) = T_t^m\psi(x)$. Hence the assumptions of Lemma 1.62 are satisfied by the uniform continuity of this function.

Lemma 1.63
Let $T > 0$, $\psi \in I^m_{PAM}$, $\psi \geq 0$ and $\gamma > 0$ be sufficiently small. It holds that the mild solution $U_t^m(\gamma\psi)$, $t \leq T$, is a strong solution $\phi_m \in C^1([0,T], C_0((0,L)^2))$, $\phi_m(t) \in Dom(\mathscr{H}_{\boldsymbol{\xi}_m})$, $0 \leq t \leq T$, to

$$\begin{cases}\partial_t\phi_m(t) = \mathscr{H}_{\boldsymbol{\xi}_m}\phi_m(t) - \frac{1}{2}\phi_m(t)^2 \text{ in } (0,T)\times(0,L)^2, \\ \phi_m(0) = \gamma\psi \text{ in } [0,L]^2, \phi_m(t) = 0 \text{ on } [0,T]\times\partial[0,L]^2.\end{cases}$$

Proof
The semigroup for $\mathscr{H}_{\boldsymbol{\xi}_m}$ is strongly continuous by an application of [CZ95, Theorem 3.17, Proposition 3.23]. The claim now follows as in [Isc86, Theorem A]. Note that Iscoe assumes that the semigroup is contractive. However, this may be dropped by the localness of the problem. Also, since we have already constructed a mild solution, there is no need for us to introduce, in their notation, the function $\tilde{g}$. Their 'mild equals strong' argument subsequently carries over to our solution.

□

It follows immediately from the variation-of-constants representation that $U^m(\psi) \leq T^m(\psi)$ for any $\psi \in \mathrm{Dom}(\mathscr{H}_{\boldsymbol{\xi}_m})$. The other natural bound should read that for any $\psi \in \mathrm{Dom}(\mathscr{H}_{\boldsymbol{\xi}_m})$, $\psi \geq 0$, $0 \leq U^m(\psi)$. This part of the proof carried out in [Isc86, Theorem A] does, at least to our knowledge, not carry over, as it implicitly uses the contractivity of the semigroup. However, we will encounter in Conjecture 1.65 an argument which should yield the non-negativity at least for sufficiently small initial conditions.

1.10 The Evolution Equation for the Killed Rough Super Brownian Motion

In this section we derive the solution theory for the 'Evolution Equation for the killed rough Super Brownian Motion' given by

$$\begin{cases} \partial_t \phi(t) = \mathscr{H}_{\boldsymbol{\xi}} \phi(t) - \frac{1}{2}\phi(t)^2 \text{ in } (0,T) \times (0,L)^2, \\ \phi(0) = \gamma\psi \text{ in } [0,L]^2, \phi(t) = 0 \text{ on } [0,T] \times \partial[0,L]^2, \end{cases}$$

with $\psi \in \mathcal{C}_{\mathfrak{d}}^{\alpha+2+\delta}$, $0 < \delta < 1-(\alpha+2)$, $\psi \geq 0$, $\psi \neq 0$ and $\gamma > 0$ sufficiently small. This construction, which we shall call the Paracontrolled Wild sum approach, is novel and builds on [Eth00, Chapter 2.1]. Note that the same arguments carry over to the mollified Evolution Equation as well and yield yet another construction in this case. We define as in Section 1.9,

$$\begin{aligned} \phi_0(t) &:= T_t(\gamma\psi), \\ \phi_{n+1} &:= \phi_0(t) + S(\phi_n, \phi_n)(t), \quad n \in \mathbb{N}_0, \end{aligned}$$

where for $f, g \in C_t \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$,

$$S(f,g)(t) := -\frac{1}{2}\int_0^t T_{t-s}(f(s)g(s))ds.$$

Lemma 1.64
Let $T > 0$, $\psi \in \mathcal{C}_{\mathfrak{d}}^{\alpha+2+\delta}$, $0 < \delta < 1-(\alpha+2)$, $\psi \geq 0$, $\psi \neq 0$ and assume $\gamma > 0$ is sufficiently small. Then there exists a variation-of-constants solution $U(\gamma\psi) \in \mathscr{L}_{\mathfrak{d},T}^{\alpha+2+\delta/8}$ to the Evolution Equation for the killed rough SBM given by

$$U_t(\gamma\psi) = T_t(\gamma\psi) - \frac{1}{2}\int_0^t T_{t-s}(U_s(\gamma\psi)^2)ds.$$

In particular, it holds that $U(\gamma\psi) \in C([0,T], \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+})$.

Proof
Let first $\psi \in \mathcal{C}_{\mathfrak{d}}^{\alpha+2+\delta}$ and $v \in C_T \mathcal{C}_{\mathfrak{d}}^{\alpha+2+\delta}$ for some $0 < \delta < 1-(\alpha+2)$. We define for $t \geq 0$,

$$V(t) := \int_0^t T_{t-s} v(s)ds.$$

Let $C_1, C_2 > 0$ be the constants appearing in Lemma 1.57. We can bound

$$\|T(\psi)\|_{\mathscr{L}^{\alpha+2+\delta/2}_{\mathfrak{d},T}} \leq (C_1 \vee 1)\exp(C_2 T)\|\psi\|_{\mathcal{C}^{\alpha+2+\delta}_{\mathfrak{d}}},$$

$$\|V\|_{C_T\mathcal{C}^{\alpha+2+\delta/2}_{\mathfrak{d}}} \leq \sup_{0\leq t\leq T}\int_0^t C_1\exp(C_2(t-s))ds\|v\|_{C_T\mathcal{C}^{\alpha+2+\delta}_{\mathfrak{d}}} \leq C_1 T\exp(C_2T)\|v\|_{C_T\mathcal{C}^{\alpha+2+\delta}_{\mathfrak{d}}}.$$

We have that

$$V(t) - V(s) = (T_{t-s} - 1)\int_0^s T_{s-r}v(r)dr + \int_s^t T_{t-r}v(r)dr.$$

It follows for $0 \leq s, t \leq T$,

$$\begin{aligned}
&\|V(t) - V(s)\|_{L^\infty} \\
&\leq C_1\exp(C_2T)(t-s)^{(\alpha+2+\delta/4)/2}\left\|\int_0^s T_{s-r}v(r)dr\right\|_{\mathcal{C}^{\alpha+2+\delta/2}_{\mathfrak{d}}} + \int_s^t \|T_{t-r}v(r)\|_{\mathcal{C}^{\alpha+2+\delta/2}_{\mathfrak{d}}}dr \\
&\leq C_1\exp(C_2T)(t-s)^{(\alpha+2+\delta/4)/2}\int_0^s C_1\exp(C_2(s-r))dr\|v\|_{C_T\mathcal{C}^{\alpha+2+\delta}_{\mathfrak{d}}} \\
&+ (t-s)C_1\exp(C_2T)\|v\|_{C_T\mathcal{C}^{\alpha+2+\delta}_{\mathfrak{d}}} \\
&\leq (C_1\exp(C_2T))^2 T(t-s)^{(\alpha+2+\delta/4)/2}\|v\|_{C_T\mathcal{C}^{\alpha+2+\delta}_{\mathfrak{d}}} + (t-s)C_1\exp(C_2T)\|v\|_{C_T\mathcal{C}^{\alpha+2+\delta}_{\mathfrak{d}}}.
\end{aligned}$$

For the second term,

$$\begin{aligned}
&(t-s)C_1\exp(C_2T)\|v\|_{C_T\mathcal{C}^{\alpha+2+\delta}_{\mathfrak{d}}} \\
&\leq C_1\exp(C_2T)\|v\|_{C_T\mathcal{C}^{\alpha+2+\delta}_{\mathfrak{d}}}(t-s)^{(\alpha+2+\delta/4)/2}(t-s)^{1-(\alpha+2+\delta/4)/2} \\
&\leq C_1\exp(C_2T)\|v\|_{C_T\mathcal{C}^{\alpha+2+\delta}_{\mathfrak{d}}}(t-s)^{(\alpha+2+\delta/4)/2}(T \vee 1).
\end{aligned}$$

This yields that $V \in C_T^{(\alpha+2+\delta/4)/2}L^\infty$ with the bound

$$\|V\|_{\mathscr{L}^{\alpha+2+\delta/4}_{\mathfrak{d},T}} \leq \max\left\{C_1\exp(C_2T)(T\vee 1), (C_1\exp(C_2T))^2T\right\}\|v\|_{C_T\mathcal{C}^{\alpha+2+\delta}_{\mathfrak{d}}}.$$

We define the trees τ as in Section 1.9, now for the PAM instead of the mollified PAM. Let $T > 0$ and $0 < \delta < 1 - (\alpha + 2)$. We define

$$\begin{aligned}
A(T) &:= \max\left\{C_1\exp(C_2T)(T\vee 1), (C_1\exp(C_2T))^2T, 1\right\}, \\
K(T) &:= \max\left\{(C_1 \vee 1)\exp(C_2T)\|\psi\|_{\mathcal{C}^{\alpha+2+\delta}_{\mathfrak{d}}}, 1\right\}.
\end{aligned}$$

We claim that for any $n \in \mathbb{N}$, $|\tau| = n$, it holds that

$$\|\tau\|_{\mathscr{L}^{\alpha+2+\delta(5/4-\sum_{k=1}^{2n}1/2^k)}_{\mathfrak{d},T}} \leq A(T)^{n-1}K(T)^n.$$

Proof of the claim: We proceed by induction. Let $n = 1$, then

$$\|\bullet\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2+\delta/2}} \leq K(T).$$

For $|\tau| = n+1$, we decompose $\tau = {}^{\tau_1}\!\vee^{\tau_2}$. Assume that without limitation of generality, $|\tau_1| \leq |\tau_2| \leq n$. It follows by the above and inductively, that

$$\begin{aligned}
&\|{}^{\tau_1}\!\vee^{\tau_2}\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2+\delta(5/4-\sum_{k=1}^{2(n+1)} 1/2^k)}} \leq \frac{1}{2}A(T)\|\tau_1\tau_2\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2+\delta(5/4-\sum_{k=1}^{2n} 1/2^k)}} \\
&\leq A(T)\|\tau_1\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2+\delta(5/4-\sum_{k=1}^{2|\tau_1|} 1/2^k)}}\|\tau_2\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2+\delta(5/4-\sum_{k=1}^{2|\tau_2|} 1/2^k)}} \\
&\leq A(T)^{|\tau_1|+|\tau_2|-1}K(T)^{|\tau_1|+|\tau_2|}.
\end{aligned}$$

This yields the claim.
Let $a_n := \sum_{\tau\in\mathcal{T},|\tau|=n} \tau$. It follows that

$$\left\|\sum_{n=1}^{\infty} \gamma^n a_n\right\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2+\delta/4}} \leq \sum_{n=1}^{\infty} \gamma^n C(n)A(T)^n K(T)^n. \tag{13}$$

By Lemma 1.61 there exist for any $T > 0$, γ sufficiently small such that the sequence $(\phi_n)_{n\in\mathbb{N}}$ is bounded in $\mathscr{L}_{\mathfrak{d},T}^{\alpha+2+\delta/4}$. By the compact embedding $\mathscr{L}_{\mathfrak{d},T}^{\alpha+2+\delta/4} \subset \mathscr{L}_{\mathfrak{d},T}^{\alpha+2+\delta/8}$, Lemma 1.33, it follows that there exists a converging subsequence, which converges to some $U(\gamma\psi) \in \mathscr{L}_{\mathfrak{d},T}^{\alpha+2+\delta/8}$ such that

$$U_t(\gamma\psi) = T_t(\gamma\psi) - \frac{1}{2}\int_0^t T_{t-s}(U_s(\gamma\psi)^2)ds.$$

□

We also need to establish natural bounds for the solutions. This is the content of the following conjecture.

Conjecture 1.65
Let $T > 0$, $\psi \in \mathcal{C}_{\mathfrak{d}}^{\alpha+2+\delta}$, $0 < \delta < 1-(\alpha+2)$, $\psi \geq 0$, $\psi \neq 0$. Assume $\gamma > 0$ is sufficiently small for $U(\gamma\psi) \in \mathscr{L}_{\mathfrak{d},T}^{\alpha+2+\delta/8}$ given by

$$U_t(\gamma\psi) = T_t(\gamma\psi) - \frac{1}{2}\int_0^t T_{t-s}(U_s(\gamma\psi)^2)ds$$

to exist by an application of Lemma 1.64. Then it holds that $0 \leq U_t(\gamma\psi) \leq T_t(\gamma\psi)$ for any $0 \leq t \leq T$.

Sketch of Proof
The inequality $U_t(\gamma\psi) \leq T_t(\gamma\psi)$ is immediate by the variation-of-constants representation.

For the lower bound assume that there exist some t, x such that $U_t(\gamma\psi)(x) < 0$. Then

$$T_t(\gamma\psi)(x) < \frac{1}{2}\int_0^t T_{t-s}(U_s(\gamma\psi)^2)(x)ds.$$

Note that $U_s(\gamma\psi)/\gamma \to T_s(\psi)$ as $\gamma \downarrow 0$. Consequently the LHS is of order γ, while the RHS is of order γ^2. This yields a contradiction for γ sufficiently small. One needs to make sure, though, that the sufficiency of γ does not depend on t, x. We give a formal argument that this should be possible:
Assume that there exists some $0 \le t \le T$ minimal such that there is some $x \in (0, L)^2$ with $U_t(\gamma\psi)(x) < 0$. If $t = 0$, then $U_0(\gamma\psi)(x) = \gamma\psi(x) \ge 0$, which yields a contradiction. Hence assume that $t > 0$. We get by using the minimality of t and the maximum principle, Lemma 1.56,

$$T_t(\gamma\psi)(x) \le \frac{1}{2}\int_0^t T_{t-s}(U_s(\gamma\psi)^2)(x)ds \le \frac{1}{2}\int_0^t T_{t-s}(U_s(\gamma\psi))(x)\|U_s(\gamma\psi)\|_{L^\infty}ds.$$

What is more,

$$T_{t-s}U_s(\gamma\psi) = T_t(\gamma\psi) - \frac{1}{2}\int_0^s T_{t-r}(U_r(\gamma\psi)^2)dr.$$

Let $s < t$. By the minimality of t, it follows that $0 \le U_s(\gamma\psi) \le T_s(\gamma\psi)$, hence $\|U_s(\gamma\psi)\|_{L^\infty} \le \|T_s(\gamma\psi)\|_{L^\infty}$. Consequently,

$$T_t(\gamma\psi)(x) \le T_t(\gamma\psi)(x)\int_0^t \|T_s(\gamma\psi)\|_{L^\infty}ds \le T_t(\gamma\psi)(x)\|\gamma\psi\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}C_1 T\exp(C_2 T), \quad (14)$$

where $C_1, C_2 > 0$ are as in Lemma 1.57. Note that $\gamma > 0$ sufficiently small in the construction of Lemma 1.64 implies in particular, that

$$\gamma < \frac{1}{4A(T)K(T)} \le \frac{1}{C_1 T\exp(C_2 T)\|\psi\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}}.$$

Therefore, (14) is a contradiction.
The reason why the argument above is formal is that there may not be such a minimal t. For example assume that for each $t > 0$, there is some $x_t \in (0, L)^2$ with $U_t(\gamma\psi)(x_t) < 0$ and $x_t \to x_0 \in \partial[0, L]^2$ as $t \downarrow 0$.

1.11 Approximation of the Rough Evolution Equation

Treating high order non-linearities is a major challenge in paracontrolled calculus. In this section, we show that $U^m(\gamma\psi) \to U(\gamma\psi)$ as $m \to \infty$, implying that the solution constructed in Section 1.10 is natural. To our knowledge, the approach used here is novel.

Lemma 1.66
Let $0 < \varepsilon < 1/4(-1-\alpha)$, $T > 0$. *Let* $\psi, \psi^m \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$ *and let* $K(\boldsymbol{\theta}), A(\boldsymbol{\theta})$, $\boldsymbol{\theta} \in \mathfrak{X}_{\mathfrak{n}}^{\alpha+4\varepsilon}$, *be the constants from Lemma 1.48. Then there exist some constants* $C > 0$ *and* $K(\boldsymbol{\theta}_k, \boldsymbol{\theta}, T) > 0$, $\boldsymbol{\theta}_k, \boldsymbol{\theta} \in \mathfrak{X}_{\mathfrak{n}}^{\alpha+4\varepsilon}$, *depending only on appropriate norms of* $\boldsymbol{\theta}_k, \boldsymbol{\theta}$, $\boldsymbol{\theta}_k - \boldsymbol{\theta}$, *with the following properties: If* $\boldsymbol{\theta}_k \to \boldsymbol{\theta}$ *in* $\mathfrak{X}_{\mathfrak{n}}^{\alpha+4\varepsilon}$ *as* $k \to \infty$, *then* $K(\boldsymbol{\theta}_k, \boldsymbol{\theta}, T) \to 0$. *Then,*

$$\begin{aligned}&\|T\psi - T^m\psi^m\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}\\ &\leq \|\psi\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} K(\boldsymbol{\xi}_m, \boldsymbol{\xi}, T)\exp(\log(K(\boldsymbol{\xi}_m, \boldsymbol{\xi}, T))(A(\boldsymbol{\xi})+1)T)\\ &+ \|\psi - \psi^m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}(K(\boldsymbol{\xi}_m)\exp(\log(K(\boldsymbol{\xi}_m))(A(\boldsymbol{\xi}_m)+1)T) + C).\end{aligned}$$

Proof
We decompose

$$\begin{aligned}&\|T\psi - T^m\psi^m\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} \leq \|T\psi - T^m\psi\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} + \|T^m\psi - T^m\psi^m\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}\\ &\leq \|T\psi - P\psi - (T^m\psi - P\psi)\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} + \|T^m\psi - P\psi - (T^m\psi^m - P\psi^m)\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}\\ &+ \|P\psi - P\psi^m\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}.\end{aligned}$$

It follows by Lemma 1.53,

$$\begin{aligned}&\|T^m\psi - P\psi - (T^m\psi^m - P\psi^m)\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}\\ &\leq \|\psi - \psi^m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} K(\boldsymbol{\xi}_m)\exp(\log(K(\boldsymbol{\xi}_m))(A(\boldsymbol{\xi}_m)+1)T).\end{aligned}$$

By Lemma 1.39 for some $C > 0$, $\|P\psi - P\psi^m\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} \leq C\|\psi - \psi^m\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}$. We have shown in Lemma 1.55, that the paracontrolled solution to the PAM admits the same structure as in the mollified case. Therefore, by repeating the derivation of Lemma 1.57 and by using bilinearity, we get for some constants as in the claim,

$$\begin{aligned}&\|(T\psi - P\psi) - (T^m\psi - P\psi)\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}\\ &\leq K(\boldsymbol{\xi}_m, \boldsymbol{\xi}, T)\exp(\log(K(\boldsymbol{\xi}_m, \boldsymbol{\xi}, T))(A(\boldsymbol{\xi})+1)T)\|\psi\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}.\end{aligned}$$

□

We can now show that solutions to the mollified Evolution Equation converge to the rough case.

Theorem 1.67
Let $T > 0$ *and* $\psi \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$, $\psi \geq 0$, $\psi \neq 0$. *Let* γ *be sufficiently small. Then it holds that in* $C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}$,

$$U^m(\gamma\psi) \to U(\gamma\psi).$$

Proof
Let $\psi \in \mathcal{C}_{\mathfrak{d}}^{\alpha+2+\delta}$, $0 < \delta < 1 - (\alpha + 2)$, be non-negative and not identically zero. We can repeat the derivation of Section 1.10 in the mollified case by using the analogous bounds. In particular, there exists a non-trivial lower bound for the sufficiency of γ. We define

$$A(T) := \max\left\{C_1 \exp(C_2 T)(T \vee 1), (C_1 \exp(C_2 T))^2 T, 1\right\},$$
$$K(T) := \max\left\{(C_1 \vee 1) \exp(C_2 T) \|\psi\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2+\delta}}, 1\right\}.$$

and

$$A_m(T) := \max\left\{C_1^m \exp(C_2^m T)(T \vee 1), (C_1^m \exp(C_2^m T))^2 T, 1\right\},$$
$$K_m(T) := \max\left\{(C_1^m \vee 1) \exp(C_2^m T) \|\psi\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2+\delta}}, 1\right\}.$$

When building trees, we use the naming convention that the superscript of the subtrees shall indicate whether we employ S^m or S:

$$S^m(\tau_1^m, \tau_2^m) =: \overset{\tau_1^m\ \tau_2^m}{\vee}, \quad S(\tau_1, \tau_2) =: \overset{\tau_1\ \tau_2}{\vee}.$$

Then by the above, for any tree τ, $\|\tau\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} \leq A(T)^{|\tau|-1} K(T)^{|\tau|}$ and for any tree τ^m, $\|\tau^m\|_{\mathscr{L}_{\mathfrak{d},T}^{\alpha+2}} \leq A_m(T)^{|\tau^m|-1} K_m(T)^{|\tau^m|}$. We define

$$A_*(T) = \max\left\{A(T), \sup_{m\in\mathbb{N}} A_m(T)\right\},$$
$$K_*(T) = \max\left\{K(T), \sup_{m\in\mathbb{N}} K_m(T)\right\},$$
$$C_*^1(T) = \max\left\{\sup_{m\in\mathbb{N}} \exp(\log(K(\boldsymbol{\xi}_m, \boldsymbol{\xi}, T))(A(\boldsymbol{\xi}) + 1)T), 1\right\},$$
$$C_*^2(T) = \max\left\{T(\sup_{m\in\mathbb{N}} K(\boldsymbol{\xi}_m) \exp(\log(K(\boldsymbol{\xi}_m))(A(\boldsymbol{\xi}_m) + 1)T) + C), 1\right\}.$$

Let τ^m be the tree obtained upon using the operator S^m and T^m instead of S and T. We claim that

$$\|\tau - \tau^m\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} \leq K(\boldsymbol{\xi}_m, \boldsymbol{\xi}, T) C_*^1(T)(T \vee 1)(2|\tau| - 1) C_*^2(T)^{|\tau|-1} A_*(T)^{|\tau|-1} K_*(T)^{|\tau|}.$$

We proceed inductively along the order n of the trees. First for $n = 1$, by Lemma 1.66,

$$\|T\psi - T^m\psi\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} \leq K(\boldsymbol{\xi}_m, \boldsymbol{\xi}, T) \exp(\log(K(\boldsymbol{\xi}_m, \boldsymbol{\xi}, T))(A(\boldsymbol{\xi}) + 1)T) \|\psi\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}.$$

Next, let $\tau_1, \tau_2, \tau_1^m, \tau_2^m$ be trees such that $|\tau_1| + |\tau_2| = n + 1$. We get

$$
\begin{aligned}
&\|\overset{\tau_1\ \tau_2}{\vee} - \overset{\tau_1^m\ \tau_2^m}{\vee}\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} \leq \frac{1}{2}\sup_{0\leq t\leq T}\int_0^t \|T(\tau_1(s)\tau_2(s)) - T^m(\tau_1^m(s)\tau_2^m(s))\|_{C_{t-s}\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} ds \\
&\leq \frac{1}{2}(T\vee 1)K(\boldsymbol{\xi}_m, \boldsymbol{\xi}, T)\exp(\log(K(\boldsymbol{\xi}_m, \boldsymbol{\xi}, T))(A(\boldsymbol{\xi}) + 1)T)\|\tau_1\tau_2\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} \\
&+ \frac{1}{2}T(K(\boldsymbol{\xi}_m)\exp(\log(K(\boldsymbol{\xi}_m))(A(\boldsymbol{\xi}_m) + 1)T) + C)\|\tau_1\tau_2 - \tau_1^m\tau_2^m\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} \\
&\leq K(\boldsymbol{\xi}_m, \boldsymbol{\xi}, T)C_*^1(T)(T\vee 1)\frac{1}{2}\|\tau_1\tau_2\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} + C_*^2(T)\frac{1}{2}\|\tau_1\tau_2 - \tau_1^m\tau_2^m\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}.
\end{aligned}
$$

It holds that

$$
\|\tau_1\tau_2\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} \leq 2\|\tau_1\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}\|\tau_2\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} \leq 2A_*(T)^{|\tau_1|+|\tau_2|-2}K_*(T)^{|\tau_1|+|\tau_2|}
$$

and

$$
\begin{aligned}
&\|\tau_1\tau_2 - \tau_1^m\tau_2^m\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} \leq 2\|\tau_1\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}\|\tau_2 - \tau_2^m\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} + 2\|\tau_2^m\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}\|\tau_1 - \tau_1^m\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} \\
&\leq 2A_*(T)^{|\tau_1|-1}K_*(T)^{|\tau_1|}\|\tau_2 - \tau_2^m\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} + 2A_*(T)^{|\tau_2^m|-1}K_*(T)^{|\tau_2^m|}\|\tau_1 - \tau_1^m\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}.
\end{aligned}
$$

Consequently, by the inductive hypothesis, using that $C_*^2(T), A_*(T) \geq 1$,

$$
\begin{aligned}
\|\overset{\tau_1\ \tau_2}{\vee} - \overset{\tau_1^m\ \tau_2^m}{\vee}\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} &\leq K(\boldsymbol{\xi}_m, \boldsymbol{\xi}, T)C_*^1(T)(T\vee 1) \\
&\times (2(|\tau_1| + |\tau_2|) - 1)C_*^2(T)^{|\tau_1|+|\tau_2|-1}A_*(T)^{|\tau_1|+|\tau_2|-1}K_*(T)^{|\tau_1|+|\tau_2|}.
\end{aligned}
$$

With this the claim is proved. Together with Lemma 1.61, this yields that

$$
\begin{aligned}
&\sum_{\tau\in\mathcal{T}}\|\tau - \tau^m\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}\gamma^{|\tau|} \\
&\leq K(\boldsymbol{\xi}_m, \boldsymbol{\xi}, T)C_*^1(T)(T\vee 1)\sum_{n=1}^{\infty}\gamma^n C(n)(2n-1)C_*^2(T)^{n-1}A_*(T)^{n-1}K_*(T)^n.
\end{aligned}
$$

The sum on the RHS is finite for γ sufficiently small, hence for $m\to\infty$,

$$
\|U(\gamma\psi) - U^m(\gamma\psi)\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}} \leq \sum_{\tau\in\mathcal{T}}\|\tau - \tau^m\|_{C_T\mathcal{C}_{\mathfrak{d}}^{\alpha+2}}\gamma^{|\tau|} \to 0.
$$

□

In particular, the same proof also yields the existence and convergence of solutions to

$$
\begin{cases}
\partial_t\phi(t) = \mathscr{H}_{\boldsymbol{\xi}}\phi(t) + \frac{1}{2}\phi(t)^2 \text{ in } (0,T)\times(0,L)^2, \\
\phi(0) = \gamma\psi \text{ in } [0,L]^2, \phi(t) = 0 \text{ on } [0,T]\times\partial[0,L]^2.
\end{cases}
$$

2 Construction of the Killed Mollified Super Brownian Motion

In this section we construct the killed mollified Super Brownian Motion as a limit of Branching Brownian Motions in a mollified white noise environment. The theory developed below is classical and the main arguments are already outlined in [Eth00, Section 1.5].

2.1 Killed Branching Brownian Motion in a Mollified White Noise Environment

First we define our population model. Then we introduce the spatial movement of particles and afterwards the branching and rescaling.

Definition 2.1
Let $n \in \mathbb{N}$, $L > 0$ and $\delta \notin \mathbb{R}^2$. Let B^n be a variance-$2/n$-Brownian Motion in $\mathbb{R}^2$ started in $x \in (0, L)^2$. Let T_n be the first exit time of $(0, L)^2$. We set

$$B_t^{L,n} = \begin{cases} B_t^n & \text{if } t < T_n, \\ \delta & \text{if } t \geq T_n, \end{cases}$$

and call it a killed variance-$2/n$-Brownian Motion.

Some properties of the killed variance-$2/n$-Brownian Motion are listed in the following lemma.

Lemma 2.2 [CZ95, Theorem 2.2, Proposition 2.3 and Proposition 3.23]
Let $B_t^{L,n}$ be a killed variance-$2/n$-Brownian Motion. Then it is a Feller- and strong Feller process in $(0, L)^2$. What is more, the transition semigroup is a strongly continuous, positive, contraction semigroup on $C_0((0, L)^2)$. On this space its generator, from here on denoted by $G_n^{\mathfrak{k}}$, acts on $Dom(G_n^{\mathfrak{k}}) = \{f | f \in C_0((0, L)^2), \Delta f \in C_0((0, L)^2)\}$ by $G_n^{\mathfrak{k}} f = 1/n \Delta f$.

Note that $\mathrm{Dom}(G_n^{\mathfrak{k}}) = \mathrm{Dom}(\mathscr{H}_{\xi_m})$. We need to consider an extension to the one-point compactification of the domain, which will allow us to test our processes against constant functions.

Lemma 2.3 [EK86, Proposition 4.2.3]
Let E be a locally compact, separable metric space. Let $(T(t))_{t \geq 0}$ be a strongly continuous, positive, contraction semigroup on $C_0(E)$ with generator G. Assume $\delta \notin E$ and let the set $E \dot{\cup} \{\delta\}$ be equipped with the topology of the one-point compactification. Define the operator $T^\delta(t)$ on $C(E \dot{\cup} \{\delta\})$ for each $t \geq 0$ by $T^\delta(t) f = f(\delta) + T(t)(f - f(\delta))$. Then $(T^\delta(t))_{t \geq 0}$ is a strongly continuous, positive, contraction semigroup on $C(E \dot{\cup} \{\delta\})$ with generator G^δ extending G such that the constant-1-function is in the bp-closure of G^δ. It

holds in this sense that $G^\delta 1 = 0$ *and* $G^\delta f = Gf$ *for any* $f \in C_0(E)$*, where we associate to* f *the extension to* $E\dot{\cup}\{\delta\}$ *given by* $f(\delta) = 0$.

Remark 2.4
Using [EK86, Proposition 4.3.1], we will not distinguish between G^δ and its bp-closure when considering martingale problems.

By applying the above theorem to $G_n^{\mathfrak{k}}$, we obtain an extension $G_n^{\mathfrak{a}}$ such that $G_n^{\mathfrak{a}} 1 = 0$. We next define the environment our particles will live in.

Definition 2.5
Let $n, m \in \mathbb{N}$. *We define on* $(0, L)^2$,

$$\zeta_{n,m}^{+1}(x) = \frac{1}{2}\left(1 + \frac{\xi_m(x) - c_m}{n}\right) \quad \textit{and} \quad \zeta_{n,m}^{-1}(x) = \frac{1}{2}\left(1 - \frac{\xi_m(x) - c_m}{n}\right).$$

We denote for n *sufficiently large the associated probability generating function on* $(0, L)^2$ *by* $\Phi_{n,m}(s) = \zeta_{n,m}^{+1} s^2 + \zeta_{n,m}^{-1}$.

Remark 2.6
For any $m \in \mathbb{N}$, ξ_m is smooth, hence bounded in $[0, L]^2$. Therefore we can choose n large enough such that $\Phi_{n,m}$ is indeed a probability generating function.

Whenever we refer to 'n sufficiently large', we associate some minimal $N_0 \in \mathbb{N}$ such that $n \geq N_0$ is required for the claim. This N_0 may change during the course of this text. This is not an issue, since we can always choose the largest such constant.
We can now define our underlying particle system.

Definition 2.7
We define $E = \{\sum_{i=1}^{k} \delta_{x_i} | k \in \mathbb{N}, x_i \in (0, L)^2 \text{ for any } i \in \{1, \ldots, k\}\}$.

Definition 2.8 [EK86, Section 9.4]
Let $n, m \in \mathbb{N}$ *with* n *large enough according Definition 2.5. We define the martingale problem for* $(L_{n,m}^y, D(L_{n,m}^y))$ *by:*

$$D(L_{n,m}^y) = \Big\{ \exp(\langle \log(g), \cdot \rangle) | g \in Dom(G_n^{\mathfrak{a}}), \|g\|_{L^\infty} \leq 1, \inf_{(0,L)^2} g > 0 \Big\},$$

where on this set the generator is given by

$$L_{n,m}^y \exp(\langle \log(g), \cdot \rangle)(\mu) = \left\langle \frac{G_n^{\mathfrak{a}} g + \Phi_{n,m}(g) - g}{g}, \mu \right\rangle \exp(\langle \log(g), \mu \rangle), \quad \mu \in E.$$

Remark 2.9
In [EK86, Section 9.4] it is further imposed that $\|g\|_{L^\infty} < 1$. This is because they treat more general probability generating functions defined in terms of infinite power series. Nevertheless, their assertions carry over to our setting with this less restrictive definition, see also Remark 2.15.

A solution to this martingale problem exists and is unique:

Theorem 2.10 [EK86, Theorem 9.4.2]
The martingale problem for $(L^y_{n,m}, D(L^y_{n,m}))$ with starting point being any $\nu \in E$ has a unique solution.

Remark 2.11
Following [EK86, pp. 400-402], the martingale problem describes a branching stopped variance-$2/n$-Brownian Motion with death rate 1 and offspring distribution given by the probability generating function. The probability of two offsprings is given by $\zeta^{+1}_{n,m}(\cdot)$, while the probability of no offspring is $\zeta^{-1}_{n,m}(\cdot)$. See also the proof of Lemma 2.14 for a different representation.
Note that there are only countably many particles alive at any time $t \geq 0$, which implies that null sets don't accumulate over all particles.

Definition 2.12
Let $n, m \in \mathbb{N}$ with n sufficiently large according to Definition 2.5. Assume that $n\mu_{n,m} \in E$ for any $n \in \mathbb{N}$. We define $X^{n,m}_t = 1/nY^{n,m}_{nt}$, where $Y^{n,m}$ is the solution to the martingale problem for $(L^y_{n,m}, D(L^y_{n,m}))$ started in $n\mu_{n,m}$. The generator of $X^{n,m}$ will be denoted by $(L_{n,m}, D(L_{n,m}))$.

With this notation, we can finally define (a posteriori) the probability space we are working on:

Definition 2.13
Let for $n, m \in \mathbb{N}$, $n \geq N_0$, $(\Omega, \mathcal{F}, (\mathcal{F}^{n,m}_t)_{t\geq 0}, \mathbb{P})$ be a complete filtered, right-continuous probability space that supports $Y^{n,m}$.

2.2 Moment Bounds and Explicit Expressions for our Individual Based Model

In this section we show that our individual based models have moments of all orders which are uniform over finite time intervals. This will be crucial for establishing relative compactness, convergence and alternative martingale problems.
We first show a comparison principle which allows us to compare the Branching Brownian Motion in an environment with a classical branching process.

Lemma 2.14
Denote by $1/n\Delta$ *the generator of a two-dimensional variance-*$2/n$*-Brownian Motion on* $\mathbb{R}^2$ *with domain* $Dom(\Delta) \subset C_0(\mathbb{R}^2)$*. Let* $Z^{n,m}$ *be the solution to the martingale problem for* $(L^z_{n,m}, D(L^z_{n,m}))$*, where* $D(L^z_{n,m}) = \{\exp(\langle \log(g), \cdot \rangle) | g \in Dom(\Delta^\delta), \inf_{\mathbb{R}^2} g > 0, \|g\|_{L^\infty} \leq 1\}$ *and where on this set the generator is given by*

$$L^z_{n,m} \exp(\langle \log(g), \cdot \rangle)(\mu) = \left\langle \frac{1/n\Delta^\delta g + \Phi_{n,m}(g) - g}{g}, \mu \right\rangle \exp(\langle \log(g), \mu \rangle),$$

with $\mu = \sum_{i=1}^k \delta_{x_i}$*,* $k \in \mathbb{N}$*,* $x_i \in \mathbb{R}^2$*,* $i \in \{1, \dots, k\}$ *and* $\Phi_{n,m}(x)$ *defined for* $x \notin [0, L]^2$ *by setting* $\xi_m(x) = c_m$*. Assume that* $Z^{n,m}_0$ *is deterministic.*
Let H^i_t *for* $i \in \mathbb{N}$ *denote the position at time* t *of the* i*-th particle to have appeared in the population. More precisely: If there are* $k > 1$ *initial particles, then we assign* $H^1, \dots, H^k$ *arbitrarily to those. If the* i*-th particle branches and this is the* j*-th branching event in the population, then we assign to its children arbitrarily the labelling* i *and* $j+k$*, where* k *is the number of initial particles. If the* i*-th particle is dead or not born yet at time* t*, we set* $H^i_t = \delta$*. We also define* t_i *to be the birth time of the* i*-th particle, i.e.* $t_i = \inf\{t \geq 0 | H^i_t \neq \delta\}$*. Then* $Z^{n,m}$ *may be represented as*

$$\begin{aligned} Z^{n,m}_t = Z^{n,m}_0 &+ \int_0^t \int_{\mathbb{N}} \int_0^1 \delta_{H^i_{s-}} \mathbb{1}_{\left\{v \leq \frac{1/2(1+(\xi_m - c_m)/n)(H^i_{s-})}{p}\right\}} N_b(dsdidv) \\ &- \int_0^t \int_{\mathbb{N}} \int_0^1 \delta_{H^i_{s-}} \mathbb{1}_{\left\{v \leq \frac{1/2(1-(\xi_m - c_m)/n)(H^i_{s-})}{1-q}\right\}} N_d(dsdidv) \\ &+ \sum_{i=1}^\infty \delta_{B^{i,H^i_t}_{t-t_i}} - \delta_{H^i_{t_i}}, \end{aligned} \tag{15}$$

where

- $p = sup_{\mathbb{R}^2} 1/2(1 + \frac{\xi_m - c_m}{n}) \in [1/2, 1)$ *for* n *sufficiently large,*
- $q = inf_{\mathbb{R}^2} 1/2(1 + \frac{\xi_m - c_m}{n}) \in (0, 1/2]$ *for* n *sufficiently large,*
- $N_b(dsdidv)$ *is a Poisson random measure with intensity* $pds \sum_{k=1}^\infty \delta_k(di)dv$*,*
- $N_d(dsdidv)$ *is a Poisson random measure with intensity* $(1-q)ds \sum_{k=1}^\infty \delta_k(di)dv$*,*
- δ_x *is the Dirac mass for* $x \in \mathbb{R}^2$ *and* $\delta_\delta = 0$*,*
- $B^{i,H^i_t}_{t-t_i}$*,* $t \geq t_i$*, is a variance-*$2/n$*-Brownian Motion started in* $H^i_{t_i}$*. We set* $B^{i,\delta}_t = \delta$ *for* $t \in \mathbb{R}$*.*

Note that we can construct the random variables H^i *apart from the initial* k *ones consecutively from the Poisson Random Measures* N_b*,* N_d *and the associated indicator functions.*

Proof
This proof was inspired by [FM04, Proposition 2.6] and [Cha06, Equation (15)] and constitutes a partial extension.
Let $g \in \mathrm{Dom}(\Delta^\delta)$ be such that $\|g\|_{L^\infty} \le 1$ and define $f_g(k, x_1, \ldots, x_k) = \prod_{i=1}^k g(x_i)$. The process $Z^{n,m}$ may be equivalently characterized by the martingale problem for $(A, D(A))$, where $D(A) = \{f_g | g \in \mathrm{Dom}(\Delta^\delta), \|g\|_{L^\infty} \le 1\}$ and

$$Af_g(k, x_1, \ldots, x_k) = \sum_{j=1}^k \left(\frac{\Delta^\delta}{n} g(x_j) + \Phi_{n,m}(g(x_j)) - g(x_j) \right) \prod_{i \neq j} g(x_i).$$

To see this, it suffices to note that $\prod_{i=1}^k g(x_i) = \exp(\langle \log(g), \mu \rangle)$ if $\mu = \sum_{i=1}^k \delta_{x_i}$ and to reintroduce the factor $g(x_j)$ in the above as a productive 1, granted that $\inf_{\mathbb{R}^2} g > 0$.
We define for $\nu \in E$, $\nu = \sum_{i=1}^k \delta_{x_i}$, $f_g(\nu) = f_g(k, x_1, \ldots, x_k)$. Let $W^{n,m}$ be the process defined in (15) with $W_0^{n,m} = \nu$ and the associated H^i, see also [FM04, Theorem 3.1]. Let $(t_j^*)_{j \in \{1, \ldots, N-1\}}$ be the jump times of $W^{n,m}$ with $N-1$ the (random) number of jumps and $t_0^* = 0$, $t_N^* = t$. We get

$$\begin{aligned}
&f_g(W_t^{n,m}) - f_g(W_0^{n,m}) = \sum_{j=1}^N f_g(W_{t_j^*}^{n,m}) - f_g(W_{t_{j-1}^*}^{n,m}) \\
&= \sum_{j=1}^N f_g(W_{t_j^*}^{n,m}) - f_g(W_{t_j^*-}^{n,m}) + f_g(W_{t_j^*-}^{n,m}) - f_g(W_{t_{j-1}^*}^{n,m}) \\
&= \int_0^t \int_{\mathbb{N}} \int_0^1 (f_g(W_{s-}^{n,m} + \delta_{H_{s-}^i}) - f_g(W_{s-}^{n,m})) \mathbb{1}_{\left\{v \le \frac{1/2(1+(\xi_m - c_m)/n)}{p}(H_{s-}^i)\right\}} N_b(dsdidv) \\
&+ \int_0^t \int_{\mathbb{N}} \int_0^1 (f_g(W_{s-}^{n,m} - \delta_{H_{s-}^i}) - f_g(W_{s-}^{n,m})) \mathbb{1}_{\left\{v \le \frac{1/2(1-(\xi_m - c_m)/n)}{1-q}(H_{s-}^i)\right\}} N_d(dsdidv) \\
&+ \sum_{j=1}^N f_g(W_{t_j^*-}^{n,m}) - f_g(W_{t_{j-1}^*}^{n,m}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\mathbb{E}(f_g(W_t^{n,m})) - f_g(W_0^{n,m}) \\
&= \int_0^t \mathbb{E}\left(\sum_{j=1}^\infty \frac{1}{2} \left(1 + \frac{\xi_m - c_m}{n}\right) (H_{s-}^j) \mathbb{1}_{H_{s-}^j \neq \delta} \prod_{\substack{i \neq j \\ H_{s-}^i \neq \delta}} g(H_{s-}^i)(g(H_{s-}^j)^2 - g(H_{s-}^j)) \right) ds \\
&+ \int_0^t \mathbb{E}\left(\sum_{j=1}^\infty \frac{1}{2} \left(1 - \frac{\xi_m - c_m}{n}\right) (H_{s-}^j) \mathbb{1}_{H_{s-}^j \neq \delta} \prod_{\substack{i \neq j \\ H_{s-}^i \neq \delta}} g(H_{s-}^i)(1 - g(H_{s-}^j)) \right) ds \\
&+ \mathbb{E}\left(\sum_{j=1}^N f_g(W_{t_j^*-}^{n,m}) - f_g(W_{t_{j-1}^*}^{n,m}) \right).
\end{aligned}$$

We decompose the last term as

$$\mathbb{E}\left(\sum_{j=1}^{N} f_g(W^{n,m}_{t^*_{j-}}) - f_g(W^{n,m}_{t^*_{j-1}})\right) = \mathbb{E}\left(f_g(W^{n,m}_t) - f_g(W^{n,m}_0)|N=1\right)\mathbb{P}(N=1)$$
$$+\mathbb{E}\left(\sum_{j=1}^{N} f_g(W^{n,m}_{t^*_{j-}}) - f_g(W^{n,m}_{t^*_{j-1}})\middle| N\geq 2\right)\mathbb{P}(N\geq 2).$$

However,

$$\mathbb{P}(N\geq 2) = \mathbb{P}(N-1\geq 1) \leq \mathbb{E}(N-1) = t \to 0,$$

as $t \to 0$. Consequently, the second term does not contribute to the generator. For the first term,

$$\mathbb{P}(N=1) = \mathbb{P}(N-1=0) = \exp(-t) \to 1$$

as $t \to 0$. We see that $\mathbb{E}\left(f_g(W^{n,m}_t) - f_g(W^{n,m}_0)|N=1\right)/t$ converges as $t \to 0$ to the generator of a $\langle 1, W^{n,m}_0\rangle$-dimensional variance-$2/n$-Brownian Motion. Hence all in all we deduce that the generator A' of $W^{n,m}$ takes the form

$$\begin{aligned}
A'f_g(\nu) &= \sum_{j=1}^{k}(g(x_j)^2 - g(x_j))\frac{1}{2}\left(1+\frac{\xi_m - c_m}{n}\right)(x_j)\prod_{i\neq j} g(x_i)\\
&+\sum_{j=1}^{k}(1-g(x_j))\frac{1}{2}\left(1-\frac{\xi_m - c_m}{n}\right)(x_j)\prod_{i\neq j} g(x_i) + \sum_{j=1}^{k}\frac{\Delta^\delta}{n} g(x_j)\prod_{i\neq j} g(x_i)\\
&=\sum_{j=1}^{k}\left(g(x_j)^2\frac{1}{2}\left(1+\frac{\xi_m - c_m}{n}\right)(x_j) + \frac{1}{2}\left(1-\frac{\xi_m - c_m}{n}\right)(x_j) - g(x_j)\right)\prod_{i\neq j} g(x_i)\\
&+\sum_{j=1}^{k}\frac{\Delta^\delta}{n} g(x_j)\prod_{i\neq j} g(x_i)\\
&= Af_g(k, x_1, \ldots, x_k).
\end{aligned}$$

All in all, we may represent $Z^{n,m}$ as in equation (15).

□

Remark 2.15
The martingale problem for $Z^{n,m}$ goes in fact slightly beyond the theory presented in [EK86, Chapter 9.4], since the branching probabilities are no longer continuous. However, we can construct the process as in (15) and show that it is indeed a solution to the martingale problem for $(L^z_{n,m}, D(L^z_{n,m}))$.

In the following, when comparing processes on $(0,L)^2\dot\cup\{\delta\}$ and $\mathbb{R}^2$, 1 denotes the constant one function in both spaces, including in particular δ in the first case, whereas $\mathbb{1}_{(0,L)^2} : (0,L)^2\dot\cup\{\delta\} \to \{0,1\}$ denotes the function which is 1 in $(0,L)^2$ and 0 on $\{\delta\}$.

We can now introduce some stochastic domination results which were already applied in [Cha06].

Lemma 2.16 Cf. [Cha06, Theorem 2]
Let $n, m \in \mathbb{N}$ with n sufficiently large according to Definition 2.5 and $n\mu_{n,m} \in E$. Let $Y^{n,m}$ be the solution to the martingale problem for $(L^y_{n,m}, D(L^y_{n,m}))$. Extend ξ_m to $\mathbb{R}^2$ as in Lemma 2.14 and let $p = sup_{\mathbb{R}^2} 1/2(1 + \frac{\xi_m - c_m}{n}) \in [1/2, 1)$, $q = inf_{\mathbb{R}^2} 1/2(1 + \frac{\xi_m - c_m}{n}) \in (0, 1/2]$ and $Z^{n,m}$ be as in Lemma 2.14. Let finally Y^p be a branching Brownian Motion as $Z^{n,m}$ but with branching mechanism $\Psi(s) = ps^2 + (1-p)$. All processes start from $n\mu_{n,m}$, suitably extended. Then it holds that

$$\langle \mathbb{1}_{(0,L)^2}, Y^{n,m} \rangle \leq \langle 1, Z^{n,m} \rangle \leq \langle 1, Y^p \rangle .$$

Proof
The first inequality is clear, since the only difference between $Y^{n,m}$ and $Z^{n,m}$ is that $Z^{n,m}$ is defined on the full space. Note that by the definitions of p, q, $1 - p \leq 1 - q$. The second inequality follows by using the explicit representation (15), bounding

$$\mathbb{1}_{\left\{v \leq \frac{1/2(1+\frac{\xi_m - c_m}{n})(H^i(Z^{n,m}_{s-}))}{p}\right\}} \leq 1, \quad \mathbb{1}_{\left\{v \leq \frac{1/2(1-\frac{\xi_m - c_m}{n})(H^i(Z^{n,m}_{s-}))}{1-q}\right\}} \geq \mathbb{1}_{\left\{v \leq \frac{1-p}{1-q}\right\}},$$

as $1/2(1 - (\xi_m - c_m)/n) = 1 - 1/2(1 + (\xi_m - c_m)/n) \geq 1 - p$, and then by subsequently redefining the random variables H^i.

□

We already get a very preliminary moment existence result:

Lemma 2.17
Let $n, m \in \mathbb{N}$ with n sufficiently large according to Definition 2.5 and let $Y^{n,m}$ be the solution to the martingale problem for $(L^y_{n,m}, D(L^y_{n,m}))$ started in $n\mu_{n,m} \in E$. Then it holds that for any $k \in \mathbb{N}$ and $t > 0$,

$$\mathbb{E}\left(\left\langle \mathbb{1}_{(0,L)^2}, Y^{n,m}_t \right\rangle^k\right) < \infty.$$

Proof
The process $\langle \mathbb{1}_{(0,L)^2}, Y^{n,m} \rangle$ can be controlled by the process $\langle 1, Y^p \rangle$. By [AN72, Corollary III.6.1] this process has moments of all orders for all positive fixed times, since for any $k \in \mathbb{N}$, $2^k p < \infty$.

□

We will also need the following more refined bound.

Lemma 2.18 [EK86, Lemma 9.4.1]
Let $n, m \in \mathbb{N}$ with n sufficiently large according to Definition 2.5 and let $Y^{n,m}$ be the solution to the martingale problem for $(L^y_{n,m}, D(L^y_{n,m}))$ started in $n\mu_{n,m} \in E$. Then

$$\mathbb{E}(\langle 1, Y^{n,m}_t \rangle) \leq \langle 1, n\mu_{n,m}\rangle \exp(t\|\Phi'_{n,m}(1) - 1\|_{L^\infty})$$

and for any $K, T > 0$,

$$\mathbb{P}\left(\sup_{t\leq T} \langle 1, Y^{n,m}_t \rangle \exp(-t\|\Phi'_{n,m}(1) - 1\|_{L^\infty}) \geq K\right) \leq \frac{\langle 1, n\mu_{n,m}\rangle}{K}.$$

Proof
The second inequality differs slightly from the formulation in [EK86, Lemma 9.4.1], hence we give a full proof.
Let $\lambda > 0$ and $g = \exp(-\lambda)$. Then by the martingale problem for $(L^y_{n,m}, D(L^y_{n,m}))$,

$$\begin{aligned} M_\lambda(t) =& \exp(-\lambda \langle 1, Y^{n,m}_t \rangle) \\ &- \int_0^t \exp(-\lambda \langle 1, Y^{n,m}_s \rangle) \langle (\Phi_{n,m}(\exp(-\lambda)) - \exp(-\lambda)) \exp(\lambda), Y^{n,m}_s \rangle \, ds, \end{aligned}$$

is a martingale started in $\exp(-\lambda\langle 1, n\mu_{n,m}\rangle)$ by the martingale problem for $(L^y_{n,m}, D(L^y_{n,m}))$. Therefore,

$$\begin{aligned} &\mathbb{E}(\exp(-\lambda \langle 1, Y^{n,m}_t \rangle)) = \exp(-\lambda \langle 1, n\mu_{n,m}\rangle) \\ &+ \int_0^t \mathbb{E}(\exp(-\lambda \langle 1, Y^{n,m}_s \rangle) \langle (\Phi_{n,m}(\exp(-\lambda)) - \exp(-\lambda)) \exp(\lambda), Y^{n,m}_s \rangle) ds \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}(\exp(-\lambda \langle 1, Y^{n,m}_t \rangle)\lambda \langle 1, Y^{n,m}_t \rangle) \leq \mathbb{E}(1 - \exp(-\lambda \langle 1, Y^{n,m}_t \rangle)) \\ &= 1 - \exp(-\lambda \langle 1, n\mu_{n,m}\rangle) \\ &+ \int_0^t \mathbb{E}(\exp(-\lambda \langle 1, Y^{n,m}_s \rangle) \langle 1 - \exp(\lambda)\Phi_{n,m}(\exp(-\lambda)), Y^{n,m}_s \rangle) ds \\ &\leq 1 - \exp(-\lambda \langle 1, n\mu_{n,m}\rangle) + \int_0^t \mathbb{E}(\exp(-\lambda \langle 1, Y^{n,m}_s \rangle)\langle (\Phi'_{n,m}(1) - 1)\lambda + o(\lambda), Y^{n,m}_s \rangle) ds \\ &\leq 1 - \exp(-\lambda \langle 1, n\mu_{n,m}\rangle) \\ &+ \int_0^t \|(\Phi'_{n,m}(1) - 1) + o(1)\|_{L^\infty} \mathbb{E}(\exp(-\lambda \langle 1, Y^{n,m}_s \rangle)\lambda \langle 1, Y^{n,m}_s \rangle) ds. \end{aligned}$$

The uniformity of the error term $o(1)$ in x follows by its explicit Lagrange (mean-value) representation and the boundedness of ξ_m on $[0, L]^2$. By Gronwall's inequality,

$$\begin{aligned} &\mathbb{E}(\exp(-\lambda \langle 1, Y^{n,m}_t \rangle) \langle 1, Y^{n,m}_t \rangle) \\ &\leq \lambda^{-1}(1 - \exp(-\lambda \langle 1, n\mu_{n,m}\rangle)) \exp(t\|(\Phi'_{n,m}(1) - 1) + o(1)\|_{L^\infty}). \end{aligned}$$

Letting $\lambda \to 0$ yields that

$$\mathbb{E}(\langle 1, Y_t^{n,m}\rangle) \leq \langle 1, n\mu_{n,m}\rangle \exp(t\|\Phi'_{n,m}(1) - 1\|_{L^\infty}).$$

Further

$$M(t) := \lim_{\lambda\to 0} \lambda^{-1}(1 - M_\lambda(t)) = \langle 1, Y_t^{n,m}\rangle - \int_0^t \langle \Phi'_{n,m}(1) - 1, Y_s^{n,m}\rangle ds.$$

The convergence holds in L^1 by the dominated convergence theorem and hence M is a martingale. By the discontinuous Itô's product rule, cf. [Pro05, p. 83], we get

$$\begin{aligned}
&\langle 1, Y_t^{n,m}\rangle \exp(-t\|\Phi'_{n,m}(1) - 1\|_{L^\infty}) - \langle 1, n\mu_{n,m}\rangle \\
&= \int_0^t \exp(-s\|\Phi'_{n,m}(1) - 1\|_{L^\infty}) dM(s) \\
&+ \int_0^t \exp(-s\|\Phi'_{n,m}(1) - 1\|_{L^\infty})\langle \Phi'_{n,m}(1) - 1, Y_s^{n,m}\rangle ds \\
&- \int_0^t \langle 1, Y_s^{n,m}\rangle \, \|\Phi'_{n,m}(1) - 1\|_{L^\infty} \exp(-s\|\Phi'_{n,m}(1) - 1\|_{L^\infty}) ds,
\end{aligned}$$

where the covariation vanishes, since $t \mapsto \exp(-t\|\Phi'_{n,m}(1) - 1\|_{L^\infty})$ is of finite variation. Using that $\langle \Phi'_{n,m}(1) - 1, X_s^{n,m}\rangle \leq \|\Phi'_{n,m}(1) - 1\|_{L^\infty} \langle 1, X_s^{n,m}\rangle$, we get that

$$\langle 1, X_t^{n,m}\rangle \exp(-t\|\Phi'_{n,m}(1) - 1\|_{L^\infty})$$

is a supermartingale. Let $K, T > 0$. By [EK86, Proposition 2.2.16], for right-continuous submartingales Z,

$$\mathbb{P}\left(\sup_{t\leq T}|Z(t)| \geq K\right) \leq \mathbb{P}\left(\sup_{t\leq T} Z(t) \geq K\right) + \mathbb{P}\left(\inf_{t\leq T} Z(t) \leq -K\right) \leq \frac{2\mathbb{E}(Z^+(T)) - \mathbb{E}(Z(0))}{K}.$$

Applying this to the non-positive submartingale $-\langle 1, Y_t^{n,m}\rangle \exp(-t\|\Phi'_{n,m}(1) - 1\|_{L^\infty})$ yields the inequality

$$\mathbb{P}\left(\sup_{t\leq T} \langle 1, Y_t^{n,m}\rangle \exp(-t\|\Phi'_{n,m}(1) - 1\|_{L^\infty}) \geq K\right) \leq \frac{\langle 1, n\mu_{n,m}\rangle}{K}.$$

□

In particular for our rescaled process $X^{n,m}$,

$$\mathbb{E}(\langle 1, X_t^{n,m}\rangle) = \mathbb{E}\left(\left\langle 1, \frac{1}{n} Y_{nt}^{n,m}\right\rangle\right) \leq \frac{1}{n}\langle 1, n\mu_{n,m}\rangle \exp\left(-nt\left\|\frac{\xi_m - c_m}{n}\right\|_{L^\infty}\right).$$

Therefore, if $\mu_{n,m}$ has bounded mass uniformly in n, for any $T > 0$, it follows that $\sup_{n\geq N_0} \sup_{0\leq t\leq T} \mathbb{E}(\langle 1, X_t^{n,m}\rangle) < \infty$. From Lemma 2.17 we also know that all moments exist and Lemma 2.18 implies that the first moment is uniformly bounded. We need to establish the uniform boundedness in time of the second, third and fourth moments as well. This is the content of the next result.

Lemma 2.19
Let $n, m \in \mathbb{N}$ with n sufficiently large according to Definition 2.5. Let $Y^{n,m}$ be the solution to the martingale problem for $(L^y_{n,m}, D(L^y_{n,m}))$ started in $\nu \in E$ such that $\langle \mathbb{1}_{(0,L)^2}, \nu \rangle = 1$. Let $T > 0$. Then it holds that

$$\sup_{0\le s\le T} \mathbb{E}\left(\left\langle \mathbb{1}_{(0,L)^2}, Y^{n,m}_s \right\rangle^4\right) < \infty.$$

In particular for n sufficiently large, $p = sup_{\mathbb{R}^2} 1/2(1 + \frac{\xi_m - c_m}{n}) \in [1/2, 1)$ and Y^p as in Lemma 2.16,

$$\mathbb{E}(\langle 1, Y^p(t)\rangle) = \exp(\lambda t), \quad \mathbb{E}(\langle 1, Y^p(t)\rangle^2) = \exp(2\lambda t)\frac{\vartheta}{\lambda} + \exp(\lambda t)\left(1 - \frac{\vartheta}{\lambda}\right),$$
$$\mathbb{E}(\langle 1, Y^p(t)\rangle^3) = \exp(3\lambda t)\frac{3\vartheta^2}{2\lambda^2} + \exp(2\lambda t)\left(-\frac{3\vartheta^2}{\lambda^2} + \frac{3\vartheta}{\lambda}\right) + \exp(\lambda t)\left(\frac{3\vartheta^2}{2\lambda^2} - \frac{3\vartheta}{\lambda} + 1\right),$$

where $\lambda = 2p - 1$ and $\vartheta = 2p$. Furthermore for the fourth moment,

$$\begin{aligned}\mathbb{E}\left(\langle 1, Y^p(t)\rangle^4\right) = &\exp(4\lambda t)\frac{3\vartheta^3}{\lambda^3} + \exp(3\lambda t)\left(-\frac{9\vartheta^3}{\lambda^3} + \frac{9\vartheta^2}{\lambda^2}\right)\\ &+ \exp(2\lambda t)\left(\frac{9\vartheta^3}{\lambda^3} - \frac{18\vartheta^2}{\lambda^2} + \frac{7\vartheta}{\lambda}\right) + \exp(\lambda t)\left(-\frac{3\vartheta^3}{\lambda^3} + \frac{9\vartheta^2}{\lambda^2} - \frac{7\vartheta}{\lambda} + 1\right).\end{aligned}$$

Proof
It suffices by Lemma 2.16 to consider the process Y^p with $p = sup_{\mathbb{R}^2} 1/2(1 + (\xi_m - c_m)/n) \in [1/2, 1)$ for n sufficiently large. Let $\Psi(s) = ps^2 + (1-p)$, $u(s) = \Psi(s) - s$ and for $s, t \ge 0$, $F(s,t) = \mathbb{E}(s^{\langle 1, Y^p(t)\rangle})$. An application of the Kolmogorov backward equation yields

$$\partial_t F(s,t) = u(F(s,t)), \quad F(s,0) = s,$$

see [AN72, pp. 102ff.] for more details. Then $\mathbb{E}(\langle 1, Y^p(t)\rangle) = \partial_s\big|_{s\uparrow 1}\mathbb{E}(s^{\langle 1, Y^p(t)\rangle})$. Exchanging differentiation and integration is valid here if $1/2 \le s \le 1$. Indeed $Y^p(t) \ge 0$ and $Y^p(t)s^{Y^p(t)} \le Y^p(t)$, which is integrable by the proof of Lemma 2.17. The corresponding validity assertions for higher moments are analogous and we will not consider them explicitly on every occasion.
We define $\lambda = u'(1)$ and $\vartheta = u''(1)$. By the Kolmogorov backward equation,

$$\partial_t\partial_s F(s,t) = \partial_s\partial_t F(s,t) = \partial_s u(F(s,t)) = u'(F(s,t))\partial_s F(s,t), \quad \partial_s F(s,0) = 1.$$

Letting $s \uparrow 1$ yields,

$$\partial_t\partial_s F(1,t) = \lambda\partial_s F(1,t), \quad \partial_s F(1,0) = 1.$$

So, $\mathbb{E}(\langle 1, Y^p(t)\rangle) = \partial_s F(1,t) = \exp(\lambda t)$. For the second moment, $\partial_s^2\big|_{s\uparrow 1} F(s,t) = \mathbb{E}(\langle 1, Y^p(t)\rangle^2) - \mathbb{E}(\langle 1, Y^p(t)\rangle)$. We compute

$$\begin{aligned}\partial_t \partial_s^2 F(s,t) = \partial_s^2 \partial_t F(s,t) &= \partial_s (u'(F(s,t))\partial_s F(s,t)) \\ &= u''(F(s,t))(\partial_s F(s,t))^2 + u'(F(s,t))\partial_s^2 F(s,t)\end{aligned}$$

and

$$\partial_s^2 F(s,0) = \mathbb{E}(\langle 1, Y^p(t)\rangle \left(\langle 1, Y^p(t)\rangle - 1\right) s^{\langle 1, Y^p(t)\rangle - 2})\big|_{t=0} = 0.$$

Letting $s \uparrow 1$ yields

$$\partial_t \partial_s^2 F(1,t) = \vartheta \exp(2\lambda t) + \lambda \partial_s^2 F(1,t), \quad \partial_s^2 F(1,0) = 0.$$

Variation of constants gives that

$$\partial_s^2 F(1,t) = \exp(2\lambda t)\frac{\vartheta}{\lambda} + \exp(\lambda t)\left(-\frac{\vartheta}{\lambda}\right).$$

Hence

$$\mathbb{E}(\langle 1, Y^p(t)\rangle^2) = \exp(2\lambda t)\frac{\vartheta}{\lambda} + \exp(\lambda t)\left(1 - \frac{\vartheta}{\lambda}\right).$$

For the third moment, $\partial_s^3\big|_{s\uparrow 1} F(s,t) = \mathbb{E}(\langle 1, Y^p(t)\rangle^3) - 3\mathbb{E}(\langle 1, Y^p(t)\rangle^2) + 2\mathbb{E}(\langle 1, Y^p(t)\rangle)$. Again,

$$\begin{aligned}\partial_t \partial_s^3 F(s,t) &= \partial_s (u''(F(s,t))(\partial_s F(s,t))^2 + u'(F(s,t))\partial_s^2 F(s,t)) \\ &= u''(F(s,t)) 3 \partial_s F(s,t) \partial_s^2 F(s,t) + u'(F(s,t))\partial_s^3 F(s,t)\end{aligned}$$

and

$$\partial_s^3 F(s,0) = \mathbb{E}(\langle 1, Y^p(t)\rangle \left(\langle 1, Y^p(t)\rangle - 1\right)(\langle 1, Y^p(t)\rangle - 2) s^{\langle 1, Y^p(t)\rangle - 3})\big|_{t=0} = 0.$$

Letting $s \uparrow 1$ yields

$$\partial_t \partial_s^3 F(1,t) = \exp(3\lambda t)\frac{3\vartheta^2}{\lambda} + \exp(2\lambda t)\left(-\frac{3\vartheta^2}{\lambda}\right) + \lambda \partial_s^3 F(1,t), \quad \partial_s^3 F(1,0) = 0.$$

Variation of constants gives that

$$\partial_s^3 F(1,t) = \exp(3\lambda t)\frac{3\vartheta^2}{2\lambda^2} + \exp(2\lambda t)\left(-\frac{3\vartheta^2}{\lambda^2}\right) + \exp(\lambda t)\frac{3\vartheta^2}{2\lambda^2}.$$

Hence,

$$\mathbb{E}(\langle 1, Y^p(t)\rangle^3) = \exp(3\lambda t)\frac{3\vartheta^2}{2\lambda^2} + \exp(2\lambda t)\left(-\frac{3\vartheta^2}{\lambda^2} + \frac{3\vartheta}{\lambda}\right) + \exp(\lambda t)\left(\frac{3\vartheta^2}{2\lambda^2} - \frac{3\vartheta}{\lambda} + 1\right).$$

This implies in particular that the first three moments are integrable over finite time horizons. We repeat the computations for the fourth moment. We have

$$\partial_s^4 F(s,t) = \mathbb{E}\left(\langle 1, Y^p(t)\rangle\,(\langle 1, Y^p(t)\rangle - 1)(\langle 1, Y^p(t)\rangle - 2)(\langle 1, Y^p(t)\rangle - 3)s^{\langle 1, Y^p(t)\rangle - 4}\right)$$

and

$$\mathbb{E}(\langle 1, Y^p(t)\rangle^4) = \partial_s^4 F(1,t)\Big|_{s\uparrow 1} + 6\mathbb{E}(\langle 1, Y^p(t)\rangle^3) - 11\mathbb{E}(\langle 1, Y^p(t)\rangle^2) + 6\mathbb{E}(\langle 1, Y^p(t)\rangle),$$

as well as $\partial_s F(s,0) = 0$. By the Kolmogorov backward equation,

$$\begin{aligned}\partial_t\partial_s^4 F(s,t) &= 3u''(F(s,t))(\partial_s^2 F(s,t))^2 + 4u''(F(s,t))\partial_s F(s,t)\partial_s^3 F(s,t)\\ &\quad + u'(F(s,t))\partial_s^4 F(s,t).\end{aligned}$$

Letting $s \uparrow 1$ yields

$$\partial_t\partial_s^4 F(1,t) = \exp(4\lambda t)\frac{9\vartheta^3}{\lambda^2} + \exp(3\lambda t)\left(-\frac{18\vartheta^3}{\lambda^2}\right) + \exp(2\lambda t)\frac{9\vartheta^3}{\lambda^2} + \lambda\partial_s^4 F(1,t).$$

Variation of constants gives that

$$\partial_s^4 F(1,t) = \exp(4\lambda t)\frac{3\vartheta^3}{\lambda^3} + \exp(3\lambda t)\left(-\frac{9\vartheta^3}{\lambda^3}\right) + \exp(2\lambda t)\frac{9\vartheta^3}{\lambda^3} + \exp(\lambda t)\left(-\frac{3\vartheta^3}{\lambda^3}\right).$$

Therefore,

$$\begin{aligned}\mathbb{E}(\langle 1, Y^p(t)\rangle^4) &= \exp(4\lambda t)\frac{3\vartheta^3}{\lambda^3} + \exp(3\lambda t)\left(-\frac{9\vartheta^3}{\lambda^3} + \frac{9\vartheta^2}{\lambda^2}\right)\\ &\quad + \exp(2\lambda t)\left(\frac{9\vartheta^3}{\lambda^3} - \frac{18\vartheta^2}{\lambda^2} + \frac{7\vartheta}{\lambda}\right) + \exp(\lambda t)\left(-\frac{3\vartheta^3}{\lambda^3} + \frac{9\vartheta^2}{\lambda^2} - \frac{7\vartheta}{\lambda} + 1\right).\end{aligned}$$

□

Lemma 2.20
Let $n, m \in \mathbb{N}$ with n sufficiently large according to Definition 2.5. Let $X^{n,m}$ be as in Definition 2.12 with starting point $\mu_{n,m}$ where $n\mu_{n,m} \in E$. Assume further that $\sup_{n\geq N_0}\langle \mathbb{1}_{(0,L)^2}, \mu_{n,m}\rangle < \infty$. Then for any $T > 0$,

$$\sup_{0\leq s\leq T}\sup_{n\geq N_0}\mathbb{E}\left(\left\langle \mathbb{1}_{(0,L)^2}, X_s^{n,m}\right\rangle^4\right) < \infty.$$

Proof
We recall that $X_t^{n,m} = 1/nY_{nt}^{n,m}$, where $Y_0^{n,m} = n\mu_{n,m}$. We can choose p in Lemma 2.14

for $n \geq N_0$ as $p_n = 1/2 + O(1/n)$. Let Y^{p_n} be as in Lemma 2.14 with $Y_0^{p_n} = n\mu_{n,m}$. Let $(Y^{p_n,k})_{k\in\{1,\dots,\langle 1,n\mu_{n,m}\rangle\}}$ be i.i.d. branching processes with one initial particle each such that $Y^{p_n} \overset{d}{=} Y^{p_n,1} + \cdots + Y^{p_n,\langle 1,n\mu_{n,m}\rangle}$. Then by Lemma 2.19,

$$\begin{aligned}\mathbb{E}\left(\left\langle \mathbb{1}_{(0,L)^2}, X_s^{n,m}\right\rangle\right) = \mathbb{E}\left(\left\langle \mathbb{1}_{(0,L)^2}, \frac{1}{n}Y_{ns}^{n,m}\right\rangle\right) &\leq \frac{1}{n}\left\langle 1, n\mu_{n,m}\right\rangle \mathbb{E}\left(\left\langle 1, Y_{ns}^{p_n,1}\right\rangle\right)\\ &= \frac{1}{n}\left\langle 1, n\mu_{n,m}\right\rangle \exp(\lambda_n ns),\end{aligned}$$

where $u_n(s) = p_n s^2 + 1 - p_n - s$, $\lambda_n = u_n'(1) = 2p_n - 1 = O(1/n)$. By the independence of $Y^{p_n,k}$ and $Y^{p_n,l}$ for $k \neq l$,

$$\begin{aligned}\mathbb{E}(\langle 1, Y_{ns}^{p_n}\rangle^2) &= \sum_{k=1}^{\langle 1,n\mu_{n,m}\rangle} \mathbb{E}\left(\left\langle 1, Y_{ns}^{p_n,k}\right\rangle^2\right) + \sum_{l\neq j}\mathbb{E}\left(\left\langle 1, Y_{ns}^{p_n,l}\right\rangle\left\langle 1, Y_{ns}^{p_n,j}\right\rangle\right)\\ &= \langle 1, n\mu_{n,m}\rangle\,\mathbb{E}\left(\left\langle 1, Y_{ns}^{p_n,1}\right\rangle^2\right) + \langle 1, n\mu_{n,m}\rangle\left(\langle 1, n\mu_{n,m}\rangle - 1\right)\mathbb{E}\left(\left\langle 1, Y_{ns}^{p_n,1}\right\rangle\right)^2\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\left(\left\langle \mathbb{1}_{(0,L)^2}, X_s^{n,m}\right\rangle^2\right) &\leq \frac{1}{n^2}\mathbb{E}\left(\langle 1, Y_{ns}^{p_n}\rangle^2\right)\\ &= \frac{1}{n^2}\langle 1, n\mu_{n,m}\rangle\left(\exp(2\lambda_n ns)\frac{\vartheta_n}{\lambda_n} + \exp(\lambda_n ns)\left(1 - \frac{\vartheta_n}{\lambda_n}\right)\right)\\ &\quad + \frac{1}{n^2}\langle 1, n\mu_{n,m}\rangle\left(\langle 1, n\mu_{n,m}\rangle - 1\right)\exp(2\lambda_n ns),\end{aligned}$$

where $\vartheta_n = u_n''(1) = 2p_n = 1 + O(1/n)$. Only the first summand necessitates some discussion, since $\lambda_n \to 0$ in the denominators. However,

$$\frac{\vartheta_n}{n\lambda_n} = \frac{1 + O(1/n)}{nO(1/n)} = O(1) + O(1/n).$$

With that, uniformity in n and $s \leq T$ as well as integrability over finite time horizons follow. For the fourth moment, we apply the multinomial theorem. We write for $k_1, \dots, k_{\langle 1,n\mu_{n,m}\rangle} \in \mathbb{N}_0$ such that $k_1 + \cdots + k_{\langle 1,n\mu_{n,m}\rangle} = 4$,

$$\binom{4}{k_1, \dots, k_{\langle 1,n\mu_{n,m}\rangle}} = \frac{4!}{k_1! \cdots k_{\langle 1,n\mu_{n,m}\rangle}!}.$$

Then it follows that

$$\mathbb{E}\left(\langle 1, Y^{p_n}(t)\rangle^4\right) = \sum_{k_1+\dots+k_{\langle 1,n\mu_{n,m}\rangle}=4} \binom{4}{k_1,\dots,k_{\langle 1,n\mu_{n,m}\rangle}} \mathbb{E}\left(\prod_{l=1}^{\langle 1,n\mu_{n,m}\rangle} \left\langle 1, Y^{p_n,l}\right\rangle^{k_l}\right)$$

$$= \langle 1, n\mu_{n,m}\rangle\, \mathbb{E}\left(\left\langle 1, Y^{p_n,1}(t)\right\rangle^4\right)$$

$$+ \langle 1, n\mu_{n,m}\rangle\left(\langle 1, n\mu_{n,m}\rangle - 1\right)4\mathbb{E}\left(\left\langle 1, Y^{p_n,1}(t)\right\rangle^3\right)\mathbb{E}\left(\left\langle 1, Y^{p_n,1}(t)\right\rangle\right)$$

$$+ 1/(2!)\,\langle 1, n\mu_{n,m}\rangle\left(\langle 1, n\mu_{n,m}\rangle - 1\right)6\mathbb{E}\left(\left\langle 1, Y^{p_n,1}\right\rangle^2\right)^2$$

$$+ 1/(4!)\,\langle 1, n\mu_{n,m}\rangle\left(\langle 1, n\mu_{n,m}\rangle - 1\right)\left(\langle 1, n\mu_{n,m}\rangle - 2\right)\left(\langle 1, n\mu_{n,m}\rangle - 3\right)24\mathbb{E}\left(\left\langle 1, Y^{p_n,1}\right\rangle\right)^4,$$

where in the last step we used that $Y^{p_n,1},\dots,Y^{p_n,\langle 1,n\mu_{n,m}\rangle}$ are i.i.d.. Instead of writing down an explicit bound for the fourth moment of $X^{n,m} = 1/nY^{n,m}_{nt}$ as above, let us give the main scaling relations which lead to uniformity in n:
The time rescaling is harmless, since nt only appears in exponents together with $\lambda_n = O(1/n)$. The only dangerous terms are the λ_n appearing in the denominators. As above we have for all summands a prefactor of n^{-4}. For the fourth moment term of $Y^{p_n,1}$, the denominators are of at most order 3. However the prefactor is of order 3 after cancelling with the n from the initial distribution in $\langle 1, n\mu_{n,m}\rangle$. Consequently after cancellation with the remaining prefactor, the denominators do not vanish as $n \to \infty$. Similarly, for the third moment term, the denominators are at most of order 2 with remaining prefactor of order 2. For the second moment term, the denominators are at most of order 2, due to taking the square of the second moment, with remaining prefactor also of order 2. For the first moment term, the prefactor cancels exactly with the initial condition and the expression for the first moment does not involve λ_n in a denominator. □

From here on we will consider a substantial number of different but related martingales. For the reader's convenience and ease of reference, we have included as Section 5.2 an Index of Martingales.

Lemma 2.21 Cf. [Eth00, Lemma 1.10]
Let $n, m \in \mathbb{N}$ with n sufficiently large according to Definition 2.5. Let $\phi \in C_c^2((0,L)^2)$, $\phi \geq 0$. Then

$$L^{\phi}_{n,m}(t) := \langle \phi, X^{n,m}_t\rangle - \langle \phi, X^{n,m}_0\rangle - \int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle\, ds$$

is a martingale with predictable quadratic variation

$$\langle L^{\phi}_{n,m}\rangle_t = \int_0^t \left\langle \frac{2}{n}\nabla\phi^T\nabla\phi + \phi^2, X^{n,m}_s\right\rangle ds.$$

Proof
The first part of this proof was inspired by [Eth00, Lemma 1.10]. Our derivation of the predictable quadratic variation is novel and the result deviates from the one in [Eth00, Lemma 1.10]. Nevertheless, it is consistent with the results of [PR19].
Let $\phi \in C_c^2((0,L)^2)$, $\phi \geq 0$, $\gamma > 0$ and $g = \exp(-\gamma\phi)$. Then $g - 1 \in C_c((0,L)^2)$ and $g \in \mathrm{Dom}(G_n^{\mathfrak{a}})$. We consider the martingale problem for $Y^{n,m}$ with such g and get for $t, u \geq 0$

$$\begin{aligned}0 = \mathbb{E}\Bigg(&\exp(-\langle \gamma\phi, Y_{t+u}^{n,m}\rangle) - \exp(-\langle \gamma\phi, Y_t^{n,m}\rangle)\\ &- \int_t^{t+u} \left\langle -\frac{\gamma}{n}\Delta\phi + \frac{\gamma^2}{n}\nabla\phi^T\nabla\phi + \frac{\Phi_{n,m}(\exp(-\gamma\phi)) - \exp(-\gamma\phi)}{\exp(-\gamma\phi)}, Y_s^{n,m}\right\rangle\\ &\times \exp(-\langle \gamma\phi, Y_s^{n,m}\rangle)ds \Bigg| \mathcal{F}_t^{n,m}\Bigg)\end{aligned}$$

We set

$$\begin{aligned}f(\gamma) &:= \exp(-\langle \gamma\phi, Y_{t+u}^{n,m}\rangle) - \exp(-\langle \gamma\phi, Y_t^{n,m}\rangle)\\ &- \int_t^{t+u} \left\langle -\frac{\gamma}{n}\Delta\phi + \frac{\gamma^2}{n}\nabla\phi^T\nabla\phi + \frac{\Phi_{n,m}(\exp(-\gamma\phi)) - \exp(-\gamma\phi)}{\exp(-\gamma\phi)}, Y_s^{n,m}\right\rangle\\ &\times \exp(-\langle \gamma\phi, Y_s^{n,m}\rangle)ds\end{aligned}$$

and

$$I_1(s) = \left\langle -\frac{\gamma}{n}\Delta\phi + \frac{\gamma^2}{n}\nabla\phi^T\nabla\phi + \frac{\Phi_{n,m}(\exp(-\gamma\phi)) - \exp(-\gamma\phi)}{\exp(-\gamma\phi)}, Y_s^{n,m}\right\rangle.$$

We get

$$\begin{aligned}&\frac{d}{d\gamma}f(\gamma) = -\langle \phi, Y_{t+u}^{n,m}\rangle \exp(-\langle \gamma\phi, Y_{t+u}^{n,m}\rangle) + \langle \phi, Y_t^{n,m}\rangle \exp(-\langle \gamma\phi, Y_t^{n,m}\rangle)\\ &- \int_t^{t+u} \Big\langle -\frac{1}{n}\Delta\phi + \frac{2\gamma}{n}\nabla\phi^T\nabla\phi + \Big[\Big(\Phi_{n,m}'(\exp(-\gamma\phi))\exp(-2\gamma\phi)(-\phi) + \exp(-2\gamma\phi)\phi\Big)\\ &- \Big(\Phi_{n,m}(\exp(-\gamma\phi)) - \exp(-\gamma\phi)\Big)\exp(-\gamma\phi)(-\phi)\Big]/(\exp(-2\gamma\phi)), Y_s^{n,m}\Big\rangle\\ &\times \exp(-\langle \gamma\phi, Y_s^{n,m}\rangle) + I_1(s)\exp(-\langle \gamma\phi, Y_s^{n,m}\rangle)(-\langle \phi, Y_s^{n,m}\rangle)ds.\end{aligned}$$

We denote

$$\begin{aligned}&I_2(s) := \Big\langle -\frac{1}{n}\Delta\phi + \frac{2\gamma}{n}\nabla\phi^T\nabla\phi + \Big[\Big(\Phi_{n,m}'(\exp(-\gamma\phi))\exp(-2\gamma\phi)(-\phi) + \exp(-2\gamma\phi)\phi\Big)\\ &- \Big(\Phi_{n,m}(\exp(-\gamma\phi)) - \exp(-\gamma\phi)\Big)\exp(-\gamma\phi)(-\phi)\Big]/(\exp(-2\gamma\phi)), Y_s^{n,m}\Big\rangle.\end{aligned}$$

Note that $\frac{d}{d\gamma}I_1(s) = I_2(s)$. Plugging in $\gamma = 0$,

$$\frac{d}{d\gamma}f(0) = -\left\langle \phi, Y^{n,m}_{t+u}\right\rangle + \left\langle \phi, Y^{n,m}_{t}\right\rangle - \int_t^{t+u} \left\langle -\frac{1}{n}\Delta\phi + (1-\Phi'_{n,m}(1))\phi, Y^{n,m}_s \right\rangle ds,$$

and we note that $\Phi'_{n,m}(1) = (1+(\xi_m - c_m)/n)$. Using Lemma 2.19 we can find a suitable majorant by factoring out all deterministic functions via $\sup_{0\leq\gamma\leq 1}\|\cdot\|_{L^\infty}$ and bounding exponentials with non-positive exponents by 1. Hence we can exchange differentiation and integration, which implies

$$0 = \mathbb{E}\Bigg(-\left\langle \phi, Y^{n,m}_{t+u}\right\rangle + \left\langle \phi, Y^{n,m}_{t}\right\rangle - \int_t^{t+u} \left\langle -\frac{1}{n}(\Delta + \xi_m - c_m)\phi, Y^{n,m}_s \right\rangle ds \Bigg| \mathcal{F}^{n,m}_t \Bigg).$$

This is the martingale property for $Y^{n,m}$. The martingale property for $X^{n,m}_t = 1/nY^{n,m}_{nt}$ follows by considering ϕ/n and an application of the substitution formula.
In order to find an expression for $\langle L^{\phi}_{n,m}\rangle_t$, we decompose the term $(L^{\phi}_{n,m}(t))^2$ and identify martingales and non-decreasing, predictable processes of finite variation. In order to find another martingale, we repeat the procedure above. We have

$$\begin{aligned}
&\frac{d^2}{d\gamma^2}f(\gamma) = \left\langle \phi, Y^{n,m}_{t+u}\right\rangle^2 \exp(-\left\langle \gamma\phi, Y^{n,m}_{t+u}\right\rangle) - \left\langle \phi, Y^{n,m}_{t}\right\rangle^2 \exp(-\left\langle \gamma\phi, Y^{n,m}_{t}\right\rangle)\\
&- \int_t^{t+u} \Big\langle \frac{2}{n}\nabla\phi^T\nabla\phi + \Big[\Big(\Phi''_{n,m}(\exp(-\gamma\phi))\exp(-3\gamma\phi)\phi^2 + \Phi'_{n,m}(\exp(-\gamma\phi))\exp(-2\gamma\phi)2\phi^2\\
&+ \exp(-2\gamma\phi)(-2\phi)(\phi) - \Phi'_{n,m}(\exp(-\gamma\phi))\exp(-2\gamma\phi)\phi^2 - \Phi_{n,m}(\exp(-\gamma\phi))\exp(-\gamma\phi)\phi^2\\
&+ \exp(-2\gamma\phi)2\phi^2\Big)\exp(-2\gamma\phi) - \Big(\Phi'_{n,m}(\exp(-\gamma\phi))\exp(-2\gamma\phi)(-\phi) + \exp(-2\gamma\phi)\phi\\
&- (\Phi_{n,m}(\exp(-\gamma\phi)) - \exp(-\gamma\phi))\exp(-\gamma\phi)(-\phi)\Big)\exp(-2\gamma\phi)(-2\phi)\Big]/\exp(-4\gamma\phi), Y^{n,m}_s\Big\rangle\\
&\times \exp(-\left\langle \gamma\phi, Y^{n,m}_s\right\rangle) + I_2(s)\exp(-\left\langle \gamma\phi, Y^{n,m}_s\right\rangle)(-\left\langle \phi, Y^{n,m}_s\right\rangle)\\
&+ I_2(s)\exp(-\left\langle \gamma\phi, Y^{n,m}_s\right\rangle)(-\left\langle \phi, Y^{n,m}_s\right\rangle) + I_1(s)\exp(-\left\langle \gamma\phi, Y^{n,m}_s\right\rangle)\left\langle \phi, Y^{n,m}_s\right\rangle^2 ds.
\end{aligned}$$

Now

$$\begin{aligned}
\frac{d^2}{d\gamma^2}f(0) &= \left\langle \phi, Y^{n,m}_{t+u}\right\rangle^2 - \left\langle \phi, Y^{n,m}_{t}\right\rangle^2\\
&\quad - \int_t^{t+u} \left\langle \frac{2}{n}\nabla\phi^T\nabla\phi + \phi^2, Y^{n,m}_s\right\rangle + \left\langle \frac{2}{n}(\Delta + \xi_m - c_m)\phi, Y^{n,m}_s\right\rangle \left\langle \phi, Y^{n,m}_s\right\rangle ds.
\end{aligned}$$

As above, this yields that

$$\begin{aligned}
M^{1,\phi}_{n,m}(t) &:= \left\langle \phi, X^{n,m}_t\right\rangle^2 - \left\langle \phi, X^{n,m}_0\right\rangle^2\\
&\quad - \int_0^t \left\langle \frac{2}{n}\nabla\phi^T\nabla\phi + \phi^2, X^{n,m}_s\right\rangle + 2\left\langle \mathscr{H}_{\xi_m}\phi, X^{n,m}_s\right\rangle \left\langle \phi, X^{n,m}_s\right\rangle ds
\end{aligned}$$

is a martingale. Another martingale is given by

$$M^{2,\phi}_{n,m}(t) := \left(\left\langle \phi, X^{n,m}_t\right\rangle - \int_0^t \left\langle \mathscr{H}_{\xi_m}\phi, X^{n,m}_s\right\rangle ds\right)\left\langle \phi, X^{n,m}_0\right\rangle.$$

We need to find one last local martingale.
Claim: The process

$$M^{3,\phi}_{n,m}(t) := 2\int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle \langle \phi, X^{n,m}_s\rangle\, ds + \left(\int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle\, ds\right)^2 \\ - 2\langle \phi, X^{n,m}_t\rangle \int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle\, ds$$

is a local martingale.
Proof of the claim: By Fubini's theorem,

$$\begin{aligned}&\int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle \int_s^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_r\rangle\, dr ds\\ &= \int_0^t\int_0^t \mathbb{1}_{[0,t]}(s)\mathbb{1}_{[s,t]}(r) \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_r\rangle\, drds\\ &= \int_0^t\int_0^r \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle\, ds\, \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_r\rangle\, dr\\ &= \int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle \int_0^s \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_r\rangle\, drds,\end{aligned}$$

where in the last line we merely renamed some variables. Therefore by adding the first and the last line of the above,

$$\left(\int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_r\rangle\, ds\right)^2 = 2\int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle \int_s^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_r\rangle\, drds.$$

We now write

$$\begin{aligned}&2\int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle \int_s^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_r\rangle\, drds\\ &= 2\int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle (\langle \phi, X^{n,m}_t\rangle - \langle \phi, X^{n,m}_s\rangle - L^{\phi}_{n,m}(t) + L^{\phi}_{n,m}(s))ds\\ &= 2\langle \phi, X^{n,m}_t\rangle \int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle\, ds - 2\int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle \langle \phi, X^{n,m}_s\rangle\, ds\\ &- 2L^{\phi}_{n,m}(t)\int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle\, ds + 2\int_0^t L^{\phi}_{n,m}(s) \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle\, ds.\end{aligned}$$

This yields

$$M^{3,\phi}_{n,m}(t) = -2L^{\phi}_{n,m}(t)\int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle\, ds + 2\int_0^t L^{\phi}_{n,m}(s) \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle\, ds.$$

By applying the discontinuous Itô product rule, we get

$$\begin{aligned}&L^{\phi}_{n,m}(t)\int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle\, ds\\ &= \int_0^t\int_0^s \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_r\rangle\, drdL^{\phi}_{n,m}(s) + \int_0^t L^{\phi}_{n,m}(s) \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle\, ds\\ &+ [L^{\phi}_{n,m}, \int_0^{\cdot} \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle\, ds]_t.\end{aligned}$$

The covariation vanishes, since the Lebesgue integral term is absolutely continuous. Therefore,

$$M^{3,\phi}_{n,m}(t) = -2\int_0^t \int_0^s \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_r\rangle \, drdL^{\phi}_{n,m}(s),$$

which is a local martingale. This proves the claim.
We now get

$$\begin{aligned}
(L^{\phi}_{n,m}(t))^2 &= \langle \phi, X^{n,m}_t\rangle^2 + \langle \phi, X^{n,m}_0\rangle^2 + \left(\int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle \, ds\right)^2 - 2\langle \phi, X^{n,m}_t\rangle \langle \phi, X^{n,m}_0\rangle \\
&+ 2\langle \phi, X^{n,m}_0\rangle \int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle \, ds - 2\langle \phi, X^{n,m}_t\rangle \int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle \, ds \\
&= M^{1,\phi}_{n,m}(t) + 2\langle \phi, X^{n,m}_0\rangle^2 + \int_0^t \left\langle \frac{2}{n}\nabla\phi^T\nabla\phi + \phi^2, X^{n,m}_s\right\rangle ds \\
&+ 2\int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle \langle \phi, X^{n,m}_s\rangle \, ds + \left(\int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle \, ds\right)^2 \\
&- 2M^{2,\phi}_{n,m}(t) - 2\langle \phi, X^{n,m}_t\rangle \int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\rangle \, ds \\
&= M^{1,\phi}_{n,m}(t) - 2M^{2,\phi}_{n,m}(t) + M^{3,\phi}_{n,m}(t) + 2\langle \phi, X^{n,m}_0\rangle^2 \\
&+ \int_0^t \left\langle \frac{2}{n}\nabla\phi^T\nabla\phi + \phi^2, X^{n,m}_s\right\rangle ds.
\end{aligned}$$

This yields that indeed

$$\langle L^{\phi}_{n,m}\rangle_t = \int_0^t \left\langle \frac{2}{n}\nabla\phi^T\nabla\phi + \phi^2, X^{n,m}_s\right\rangle ds.$$

□

2.3 The Superprocess Limit: Killed Mollified Super Brownian Motion

We show in this section that $X^{n,m}$ converges for fixed m as $n \to \infty$ to an intermediate process which can be seen as a killed Super Brownian Motion in a mollified white noise environment (killed mollified SBM). In fact, the techniques in this section are standard and do not require any knowledge of Paracontrolled Calculus.
The following theorem asserts that for the construction of limiting processes it is sufficient to prove convergence of the generators, relative compactness of the prelimiting processes and uniqueness of the limit.

Theorem 2.22 Cf. [EK86, Theorem 4.8.10]
Let (E, r) be a complete and separable metric space and $\mathcal{M}(E)$ be the set of Borel measurable functions. Let $\mathcal{A} \subset \mathcal{M}(E)$, $\mu \in E$ and $L : \mathcal{A} \to \mathcal{M}(E)$ be a linear operator. Suppose that the $D([0,\infty), E)$-martingale problem for $(L, \mathcal{A})$ with starting point μ has

at most one solution. Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be a complete probability space. Let* $(\mathcal{F}_t^n)_{t\geq 0}$ *be a complete filtration for any* $n \in \mathbb{N}$*. Suppose* $(Z_t^n)_{t\geq 0}$ *is an* $(\mathcal{F}_t^n)_{t\geq 0}$*-adapted process with sample paths in* $D([0,\infty), E)$*. Assume further that* $\lim_{n\to\infty} \mathbb{P} \circ (Z_0^n)^{-1} = \delta_\mu$ *weakly. Assume that every subsequence of* $(Z^n)_{n\in\mathbb{N}}$ *has a subsubsequence* $(Z_l')_{l\in\mathbb{N}}$ *which converges weakly to some limit point* Z' *such that for any* $f \in \mathcal{A}$ *and* $t, s \geq 0$*,*

$$\begin{aligned} &f(Z_l'(t+s)) - f(Z_l'(t)) - \int_t^{t+s} Lf(Z_l'(u))du, \quad l \in \mathbb{N}, \\ &f(Z'(t+s)) - f(Z'(t)) - \int_t^{t+s} Lf(Z'(u))du, \end{aligned} \tag{16}$$

are $L^1(\mathbb{P})$*-integrable. For the sake of notation we denote the underlying probability measure associated to* Z' *again by* $\mathbb{P}$*. Assume further, that*

$$\begin{aligned} &\lim_{l\to\infty} \mathbb{E}\left(\left(f(Z_l'(t+s)) - f(Z_l'(t)) - \int_t^{t+s} Lf(Z_l'(u))du\right)\prod_{i=1}^k h_i(Z_l'(t_i))\right) \\ &= \mathbb{E}\left(\left(f(Z'(t+s)) - f(Z'(t)) - \int_t^{t+s} Lf(Z'(u))du\right)\prod_{i=1}^k h_i(Z'(t_i))\right), \end{aligned} \tag{17}$$

for any $k \geq 0$*,* $0 \leq t_1 < t_2 < \cdots < t_k \leq t < t+s$ *and* $h_1, \ldots, h_k \in C_b(E)$*, and that*

$$\lim_{l\to\infty} \mathbb{E}\left(\left(f(Z_l'(t+s)) - f(Z_l'(t)) - \int_t^{t+s} Lf(Z_l'(u))du\right)\prod_{i=1}^k h_i(Z_l'(t_i))\right) = 0. \tag{18}$$

Then there exists a càdlàg process Z*, adapted to the natural filtration, which solves the martingale problem for* $(L, \mathcal{A})$ *with initial distribution* μ*. Moreover,* $Z^n \to Z$ *in distribution in* $D([0,\infty), E)$*.*

Remark 2.23
Note that the existence of a converging subsubsequence for any subsequence is given if $(\mathrm{Law}(Z^n))_{n\in\mathbb{N}}$ is relatively compact in $M_1(D([0,\infty), E))$. The point is that we have to find one, which also satisfies (16), (17) and (18).

Proof of Theorem 2.22
Take by the relative compactness of $(Z^n)_{n\in\mathbb{N}}$, Z' to be a weak limit point of a converging subsubsequence $(Z_l')_{l\in\mathbb{N}}$ as in the claim. It follows by (17) and (18), that

$$\mathbb{E}\left(\left(f(Z'(t+s)) - f(Z'(t)) - \int_t^{t+s} Lf(Z'(u))du\right)\prod_{i=1}^k h_i(Z'(t_i))\right) = 0. \tag{19}$$

This implies by a Dynkin class argument, that Z' indeed solves the martingale problem for $(L, \mathcal{A})$ starting in μ. The uniqueness of limit points then also implies the weak convergence of $(Z^n)_{n\in\mathbb{N}}$ itself.

□

Let us fix a state space: The space $M((0,L)^2)$ of Radon measures on $(0,L)^2$ equipped with the vague topology in duality with $C_c((0,L)^2)$-functions:

Definition 2.24
We denote the set of Radon measures on $(0, L)^2$ *by* $M((0, L)^2)$ *and the subset of finite Radon measures on* $(0, L)^2$ *by* $M_F((0, L)^2)$.

Since $(0, L)^2$ is Polish, we may define $M((0, L)^2)$ as the space of locally finite Borel measures, see [Bog07, Theorem 7.1.7].

Theorem 2.25 [Kal83, 15.7.7]
The set of Radon measures on $(0, L)^2$, $M((0, L)^2)$, *equipped with the vague topology in duality with* $C_c((0, L)^2)$*-functions, is Polish. In fact, one metrization of the* C_c*-vague topology is given by*

$$\rho(\mu, \mu') = \sum_{k=1}^{\infty} 2^{-k}(1 - \exp(-|\langle f_k, \mu\rangle - \langle f_k, \mu'\rangle|))$$

with $(f_k)_{k\in\mathbb{N}}$ *some particular sequence of functions such that* $f_k \in C_c^\infty((0, L)^2)$ *and* $0 \le f_k \le 1$ *for any* $k \in \mathbb{N}$. *One such sequence of functions may be constructed as approximations of the indicator functions* $\mathbb{1}_B$ *with* B *a finite union of balls* $B(p, q), p \in (0, L)^2 \cap (\mathbb{Q}_+ \times \mathbb{Q}_+), q \in \mathbb{Q}_+, B(p, q) \subset (0, L)^2$, $Dist(B(p, q), \partial(0, L)^2) > 0$.

Proof
The basic claim has been proved in [Kal83, 15.7.7], we merely point out some adjustments.
Denote by $f^n \in C_c((0, L)^2)$, $0 \le f^n \le 1$, $n \in \mathbb{N}$, the functions considered in [Kal83, 15.7.7]. Note that $f^n \uparrow \mathbb{1}_B$ as $n \to \infty$ for some finite union of balls B. Let $\psi \in C_c^\infty((0, L)^2)$, $\psi \ge 0$, $\int_{(0,L)^2} \psi(x)dx = 1$ and define $\psi_k = k^2\psi(kx)$. We have for some fixed n and $k \in \mathbb{N}$ sufficiently large, $\mathrm{Supp}(\psi_k * f^n) \subset \mathrm{Supp}(f^n) + 1/k\mathrm{Supp}(\psi) \subset (0, L)^2$ and $\|\psi_k * f^n\|_{L^\infty} \le \|\psi_k\|_{L^1}\|f^n\|_{L^\infty} \le 1$. It follows that $\psi_k * f^n \to f^n$ uniformly as $k \to \infty$. Consequently we can find some sequence $\psi_{k(n)} * f^n$, $n \in \mathbb{N}$, which converges pointwise and uniformly bounded to $\mathbb{1}_B$. The rest of claim follows as in [Kal83, 15.7.7].
□

The reason why we need to use the vague topology is that individuals are killed at the boundary. Next, we note that the space of càdlàg functions taking values in $M((0, L)^2)$ is Polish as well.

Lemma 2.26
The space $D([0, \infty), M((0, L)^2))$ *equipped with the Skorokhod topology is Polish.*

Proof
The assertion is a consequence of Theorem 2.25 and [EK86, Theorem 3.5.6].
□

We will need to quantify the convergence of such objects. For this, the following will be helpful:

Lemma 2.27 [EK86, Proposition 3.5.3, Remark 3.5.4]
Let (E, r) be a metric space. Let $(x_n)_{n\in\mathbb{N}}$ and x be such that $x_n \in D([0,\infty), E)$ and $x \in D([0,\infty), E)$. Then the following are equivalent:

1. *It holds that $x_n \to x$ in $D([0,\infty), E)$.*
2. *For any $T > 0$ there exists some $(\lambda_n)_{n\in\mathbb{N}}$ with each $\lambda_n : [0,\infty) \to [0,\infty)$ strictly increasing and surjective such that*
$$\lim_{n\to\infty} \sup_{0\le t\le \lambda_n(T)} |\lambda_n^{-1}(t) - t| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \sup_{0\le t\le \lambda_n(T)} r(x_n(\lambda_n^{-1}(t)), x(t)) = 0.$$
3. *For any $T > 0$ there exists some $(\lambda_n)_{n\in\mathbb{N}}$ with each $\lambda_n : [0,\infty) \to [0,\infty)$ strictly increasing and surjective such that*
$$\lim_{n\to\infty} \sup_{0\le t\le T} |\lambda_n(t) - t| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \sup_{0\le t\le T} r(x_n(t), x(\lambda_n(t))) = 0.$$

Theorem 2.28
Let (E, r) be a separable metric space. Let $(X^n)_{n\in\mathbb{N}}$ and X be processes in $D([0,\infty), E)$. Assume $X^n \to X$ weakly and that X is a.s. continuous. Let $(Z^n)_{n\in\mathbb{N}}$, Z be their counterparts from the Skorokhod representation theorem, cf. [EK86, Theorem 3.1.8]. Then a.s. $Z^n(t) \to Z(t)$ for any $t > 0$.

Proof
The important fact is that we need to make sure that the exceptional null set does not depend on the choice of t.
By choosing $T = t$ in Lemma 2.27, it follows a.s.: For any $t \ge 0$, there exist some $\lambda_n : [0,\infty) \to [0,\infty)$ strictly increasing and surjective, such that $r(Z^n(t), Z(\lambda_n(t))) \to 0$ and $\lambda_n(t) \to t$. By the a.s. continuity of Z it follows that, $r(Z(\lambda_n(t)), Z(t)) \to 0$ for any $t \ge 0$.

□

Prokhorov's theorem subsequently also implies that the notions of relative compactness and tightness are equivalent for distributions on $D([0,\infty), M((0,L)^2))$, hence we will use them interchangeably.
Our first goal is to prove tightness. For this we need a number of preliminary results.

Theorem 2.29 (Jakubowski's criterion)[DMS93, Theorem 3.6.4]
Let (E, r) be a Polish space and $(Z^n)_{n\in\mathbb{N}}$ be a family of processes in the Skorokhod space $D([0,\infty), E)$. Assume that the compact containment condition holds, i.e. for any $\eta > 0$ and $T > 0$ there exists a compact set $\Gamma_{\eta,T} \subset E$ such that :
$$\inf_{n\in\mathbb{N}} \mathbb{P}(Z^n_t \in \Gamma_{\eta,T} \textit{ for any } t \in [0,T]) \ge 1 - \eta.$$
Let $H \subset C(E)$ be separating points in E and assume it is closed under addition. Then $(Z^n)_{n\in\mathbb{N}}$ is relatively compact if and only if $(f \circ Z^n)_{n\in\mathbb{N}}$ is relatively compact for any $f \in H$.

The following result answers some further relevant questions regarding the topology of the state space:

Lemma 2.30 [Kal83, 15.7.5]
We equip $M((0,L)^2)$ with the C_c-vague topology. Let $K>0$, then it holds that the set $B=\{\mu\in M((0,L)^2)|\langle \mathbb{1}_{(0,L)^2},\mu\rangle<K\}$ is C_c-vaguely relatively compact.

The following will be useful for verifying relative compactness of the evaluated processes:

Theorem 2.31 [EK86, Theorem 3.8.6, Theorem 3.8.8]
Let (E,r) be a complete and separable metric space and $q=r\wedge 1$. Let $(Z^n)_{n\in\mathbb{N}}$ be a family of processes in $D([0,\infty),E)$. Assume that for every $\eta>0$ and $t>0$ rational there exists some $\Gamma_\eta^t\subset E$ compact such that

$$\inf_{n\in\mathbb{N}}\mathbb{P}(Z_t^n\in\Gamma_\eta^t)\geq 1-\eta. \tag{20}$$

Assume also that for any $T>0$ there exists some $\beta>1$, $C>0$, $\theta>1$ such that

$$\lim_{\delta\to 0}\sup_{n\in\mathbb{N}}\mathbb{E}(q^\beta(Z_\delta^n,Z_0^n))=0$$

and for any $n\in\mathbb{N}$, $0\leq t\leq T+1$, $0\leq u\leq t\wedge 1$,

$$\mathbb{E}(q^{\beta/2}(Z_{t+u}^n,Z_t^n)q^{\beta/2}(Z_t^n,Z_{t-u}^n))\leq Cu^\theta.$$

Then $(Z^n)_{n\in\mathbb{N}}$ is relatively compact.

Remark 2.32
Note that the compact containment condition implies (20). The strength of this result lies in the fact that it does not rely on uniform in time estimates. This fact will play an important role in Lemma 3.6.
The restriction $u\leq 1$ is not present in the formulation of [EK86, Theorem 3.8.8] but may be assumed as the proof makes no use of $u>1$.

We can now show the tightness of the processes $(X^{n,m})_{n\in\mathbb{N}}$.

Lemma 2.33
Assume that $n\mu_{n,m}\in E$, such that $\sup_{n\geq N_0}\langle \mathbb{1}_{(0,L)^2},\mu_{n,m}\rangle<\infty$. Then it follows that the processes $(X^{n,m})_{n\geq N_0}$ started in $(\mu_{n,m})_{n\geq N_0}$ are tight in $D([0,\infty),M((0,L)^2))$.

Proof
Let $\eta,T>0$. For $K>0$ we define the set $\Gamma_K=\{\mu\in M((0,L)^2)|\langle \mathbb{1}_{(0,L)^2},\mu\rangle<K\}$, which is vaguely relatively compact by Lemma 2.30. It is

$$\mathbb{P}(X_t^{n,m}\in\Gamma_K \text{ for any } t\in[0,T])=\mathbb{P}(\langle \mathbb{1}_{(0,L)^2},X_t^{n,m}\rangle<K \text{ for any } t\in[0,T]).$$

For the complementary event, as a consequence of Lemma 2.18,

$$\mathbb{P}\left(\sup_{0\leq t\leq T}\left\langle \mathbb{1}_{(0,L)^2}, X_t^{n,m}\right\rangle \geq K\right) \leq \mathbb{P}\left(\sup_{0\leq t\leq T}\langle 1, X_t^{n,m}\rangle \geq K\right)$$
$$\leq \mathbb{P}\left(\sup_{0\leq t\leq T}\langle 1, X_t^{n,m}\rangle \exp(-nt\|\Phi'_{n,m}(1)-1\|_{L^\infty}) \geq K \inf_{0\leq s\leq T}\exp(-ns\|\Phi'_{n,m}(1)-1\|_{L^\infty})\right)$$
$$\leq \frac{\langle 1, n\mu_{n,m}\rangle}{nK\inf_{0\leq s\leq T}\exp(-ns\|\Phi'_{n,m}(1)-1\|_{L^\infty}))}.$$

As $\Phi'_{n,m}(1)-1=(\xi_m-c_m)/n$, this yields an estimate which is uniform in $n\geq N_0$. Choosing K large enough depending on η and T yields uniformly in n,

$$\mathbb{P}(X_t^{n,m}\in\Gamma_K \text{ for any } t\in[0,T])\geq 1-\eta,$$

which implies that the compact containment condition is satisfied. By Jakubowski's criterion, Theorem 2.29, it is sufficient to establish tightness of the evaluated processes $(\langle\phi, X^{n,m}\rangle)_{n\geq N_0}$ for $\phi\in C_c^2((0,L)^2)$, $\phi\geq 0$.
We apply the tightness criterion of Theorem 2.31. We compute for $0<\delta<1$, using the martingales $M_{n,m}^{1,\phi}$ and $L_{n,m}^{\phi}$ of the proof of Lemma 2.21,

$$\begin{aligned}
&\mathbb{E}((\langle\phi, X_\delta^{n,m}\rangle - \langle\phi, X_0^{n,m}\rangle)^2)\\
&= \mathbb{E}(\langle\phi, X_\delta^{n,m}\rangle^2 - 2\langle\phi, X_\delta^{n,m}\rangle\langle\phi, X_0^{n,m}\rangle + \langle\phi, X_0^{n,m}\rangle^2)\\
&= \langle\phi,\mu_{n,m}\rangle^2 + \int_0^\delta \mathbb{E}\left(\left\langle \frac{2}{n}\nabla\phi^T\nabla\phi+\phi^2, X_s^{n,m}\right\rangle + 2\langle\mathscr{H}_{\boldsymbol{\xi}_m}\phi, X_s^{n,m}\rangle\langle\phi, X_s^{n,m}\rangle\right)ds\\
&+\langle\phi,\mu_{n,m}\rangle^2 - 2\langle\phi,\mu_{n,m}\rangle\left(\langle\phi,\mu_{n,m}\rangle + \int_0^\delta \mathbb{E}(\langle\mathscr{H}_{\boldsymbol{\xi}_m}\phi, X_s^{n,m}\rangle)ds\right).
\end{aligned}$$

Note that in Theorem 2.31 we can control higher exponents of q by lower ones since $q\leq 1$. Using that the first two moments are bounded uniformly in n and $s\leq 1$, we get that this term is of order δ.
More generally, we get for $t,u\geq 0$,

$$\begin{aligned}
&\mathbb{E}((\langle\phi, X_{t+u}^{n,m}\rangle - \langle\phi, X_t^{n,m}\rangle)^2\,|\mathcal{F}_t^{n,m})\\
&= \mathbb{E}\left(\int_t^{t+u}\left\langle\frac{2}{n}\nabla\phi^T\nabla\phi+\phi^2, X_s^{n,m}\right\rangle + 2\langle\mathscr{H}_{\boldsymbol{\xi}_m}\phi, X_s^{n,m}\rangle\langle\phi, X_s^{n,m}\rangle\,ds\Big|\mathcal{F}_t^{n,m}\right)\\
&-2\mathbb{E}\left(\int_t^{t+u}\langle\mathscr{H}_{\boldsymbol{\xi}_m}\phi, X_s^{n,m}\rangle\,ds\,\langle\phi, X_t^{n,m}\rangle\,\Big|\mathcal{F}_t^{n,m}\right).
\end{aligned} \tag{21}$$

With this we can compute with Hölder's inequality,

$$
\begin{aligned}
&\mathbb{E}\left((|\langle\phi, X^{n,m}_{t+u}\rangle - \langle\phi, X^{n,m}_{t}\rangle| \wedge 1)^2\,(|\langle\phi, X^{n,m}_{t}\rangle - \langle\phi, X^{n,m}_{t-u}\rangle| \wedge 1)^2\right)\\
&\leq \mathbb{E}\left(\mathbb{E}\left((\langle\phi, X^{n,m}_{t+u}\rangle - \langle\phi, X^{n,m}_{t}\rangle)^2 \,\Big|\mathcal{F}^{n,m}_t\right)|\langle\phi, X^{n,m}_{t}\rangle - \langle\phi, X^{n,m}_{t-u}\rangle|\right)\\
&\lesssim \mathbb{E}\left(\mathbb{E}\left(\int_t^{t+u}\left\langle\frac{2}{n}\nabla\phi^T\nabla\phi + \phi^2, X^{n,m}_s\right\rangle + 2\left\langle\mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\right\rangle\left\langle\phi, X^{n,m}_s\right\rangle ds\Big|\mathcal{F}^{n,m}_t\right)^2\right)^{1/2}\\
&\times \mathbb{E}\left((\langle\phi, X^{n,m}_{t}\rangle - \langle\phi, X^{n,m}_{t-u}\rangle)^2\right)^{1/2} \qquad (22)\\
&+\mathbb{E}\left(\mathbb{E}\left(\int_t^{t+u}\left\langle\mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\right\rangle ds\left\langle\phi, X^{n,m}_t\right\rangle\Big|\mathcal{F}^{n,m}_t\right)^2\right)^{1/2}\\
&\times \mathbb{E}\left((\langle\phi, X^{n,m}_{t}\rangle - \langle\phi, X^{n,m}_{t-u}\rangle)^2\right)^{1/2}.
\end{aligned}
$$

For the first term by Jensen's inequality,

$$
\begin{aligned}
&\mathbb{E}\left(\mathbb{E}\left(\int_t^{t+u}\left\langle\frac{2}{n}\nabla\phi^T\nabla\phi + \phi^2, X^{n,m}_s\right\rangle + 2\left\langle\mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\right\rangle\left\langle\phi, X^{n,m}_s\right\rangle ds\Big|\mathcal{F}^{n,m}_t\right)^2\right)^{1/2}\\
&\leq \left(u\int_t^{t+u}\mathbb{E}\left(\left(\left\langle\frac{2}{n}\nabla\phi^T\nabla\phi + \phi^2, X^{n,m}_s\right\rangle + 2\left\langle\mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\right\rangle\left\langle\phi, X^{n,m}_s\right\rangle\right)^2\right)ds\right)^{1/2}.
\end{aligned}
$$

The functions in the integral can be bounded uniformly in n by Lemma 2.20. So, this term is of order u uniformly in n. Similar considerations apply to the term

$$
\mathbb{E}\left(\mathbb{E}\left(\int_t^{t+u}\left\langle\mathscr{H}_{\boldsymbol{\xi}_m}\phi, X^{n,m}_s\right\rangle ds\left\langle\phi, X^{n,m}_t\right\rangle\Big|\mathcal{F}^{n,m}_t\right)^2\right)^{1/2},
$$

which is also of order u. By taking the expectation of (21), we establish that

$$
\mathbb{E}\left((\langle\phi, X^{n,m}_{t}\rangle - \langle\phi, X^{n,m}_{t-u}\rangle)^2\right)^{1/2} \lesssim u^{1/2}
$$

and that both summands in (22) are of a sufficiently high order. This yields the tightness of $(X^{n,m})_{n\geq N_0}$.

□

The martingale problem for the killed mollified Super Brownian Motion is given by:

Definition 2.34
We define the martingale problem for $(L_m, D(L_m))$ by:

$$
D(L_m) = \{\exp(-\langle\phi,\cdot\rangle)|\phi \in Dom(\mathscr{H}_{\boldsymbol{\xi}_m}), \phi \geq 0\},
$$

where for $\phi \in Dom(\mathscr{H}_{\boldsymbol{\xi}_m})$, $\phi \geq 0$, the generator is given by

$$
L_m\exp(-\langle\phi,\cdot\rangle)(\mu) = \left\langle -\mathscr{H}_{\boldsymbol{\xi}_m}\phi + \frac{1}{2}\phi^2, \mu\right\rangle\exp(-\langle\phi,\mu\rangle).
$$

We call any solution to this martingale problem a killed mollified Super Brownian Motion.

The following is a technical result which will be applied in the next theorem.

Lemma 2.35
Let $\phi \in C_0((0,L)^2)$ and extend any such function by 0 to $[0,L]^2$. Then it holds that for n sufficiently large, $n\log(1-\phi/n) \to -\phi$ uniformly in $[0,L]^2$.

Proof
Dini's theorem states that it suffices to show that $n \mapsto n\log(1-\phi/n)$ is non-decreasing and converges pointwise to $-\phi$. Let $N_0 \in \mathbb{N}$ be such that for any $n \geq N_0$, $\phi(x)/n < 1$ for any $x \in [0,L]^2$. Pointwise convergence is clear, since $n\log(1-\phi/n) = \log((1-\phi/n)^n)$ and it suffices to show monotonicity. If we interpret $[N_0,\infty) \ni n \mapsto n\log(1-\phi/n)$ as a function with continuous domain, then we get

$$\frac{d}{dn} n\log\left(1-\frac{\phi(x)}{n}\right) = \log\left(1-\frac{\phi(x)}{n}\right) + \frac{\phi(x)}{n}\frac{1}{1-\frac{\phi(x)}{n}}.$$

It suffices to show that this derivative stays non-negative if n is large enough. We write for $0 \leq y \leq 1$, $f(y) = \log(1-y) + y/(1-y)$. For $y = 0$, $f(0) = 0$. Next for $y \in (0,1)$,

$$\frac{d}{dy} f(y) = \frac{y}{(1-y)^2} \geq 0.$$

Noting that $y = \phi(x)/n$ yields the claim.

□

We also need to extend the space of admissible functions in the definition of C_c-vague convergence.

Lemma 2.36
Assume that $\mu_n \in M((0,L)^2)$, $n \in \mathbb{N}$, such that $\mu_n \to \mu$, C_c-vaguely. Then, if $K := \sup_{n\in\mathbb{N}}\langle \mathbb{1}_{(0,L)^2}, \mu_n\rangle < \infty$, it follows that $\mu_n \to \mu$, C_0-vaguely, i.e. in duality with $C_0((0,L)^2)$-functions.

Proof
We first show that $\langle \mathbb{1}_{(0,L)^2}, \mu\rangle < \infty$. To see this, let $\psi_k \uparrow \mathbb{1}_{(0,L)^2}$, $\psi_k \in C_c((0,L)^2)$, $0 \leq \psi_k \leq 1$, $k \in \mathbb{N}$. Then for k and $\varepsilon > 0$ fixed, $|\langle \psi_k, \mu_n - \mu\rangle| \leq \varepsilon$ for n sufficiently large by the definition of C_c-vague convergence. Also, $|\langle \psi_k, \mu_n\rangle| \leq \|\psi_k\|_{L^\infty}\langle \mathbb{1}_{(0,L)^2}, \mu_n\rangle \leq K < \infty$, uniformly in n and k. Consequently uniformly in $k \in \mathbb{N}$, $|\langle \psi_k, \mu\rangle| \leq \varepsilon + K$. The monotone convergence theorem then yields the claim. Let $\phi \in C_0((0,L)^2)$. By the Stone-Weierstrass theorem for locally compact spaces, it follows that there exist some

$(\phi_k)_{k\in\mathbb{N}}$ such that $\phi_k \in C_c((0,L)^2)$ for any $k \in \mathbb{N}$ and $\phi_k \to \phi$ uniformly. We write

$$\begin{aligned}
&|\langle \phi, \mu_n\rangle - \langle \phi, \mu\rangle| \\
&\le |\langle \phi, \mu_n\rangle - \langle \phi_k, \mu_n\rangle| + |\langle \phi_k, \mu_n\rangle - \langle \phi_k, \mu\rangle| + |\langle \phi_k, \mu\rangle - \langle \phi, \mu\rangle| \\
&\le \sup_{n\in\mathbb{N}} \langle \mathbb{1}_{(0,L)^2}, \mu_n\rangle \|\phi - \phi_k\|_{L^\infty} + |\langle \phi_k, \mu_n\rangle - \langle \phi_k, \mu\rangle| + \langle \mathbb{1}_{(0,L)^2}, \mu\rangle \|\phi - \phi_k\|_{L^\infty}.
\end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. For the first and the third term we can choose k large enough independently of n such that those are bounded by $\varepsilon/3$. For the third term we can then choose n large enough by the definition of C_c-vague convergence to also bound this term by $\varepsilon/3$. This yields the claim.

□

Following the approach from [Eth00, Section 1.5], we can now deduce convergence of the generators:

Theorem 2.37 Cf. [Eth00, Section 1.5]
Let $m \in \mathbb{N}$, $n\mu_{n,m} \in E$, $n \in \mathbb{N}$, and $\mu_m \in M_F((0,L)^2)$ such that $\mu_{n,m} \to \mu_m$ in $M((0,L)^2)$ and also $\sup_{n\ge N_0}\langle \mathbb{1}_{(0,L)^2}, \mu_{n,m}\rangle < \infty$. Then conditions (16), (17) *and* (18) *of Theorem 2.22 hold for the martingale problems for $(L_{n,m}, D(L_{n,m}))_{n\ge N_0}$ and $(L_m, D(L_m))$ started in $\mu_{n,m}$ and $\mu_m \in M_F((0,L)^2)$ respectively.*

Proof
It is important to note that the martingale problem for $(L_m, D(L_m))$ treats functions vanishing at the boundary, while we equipped the state space with the topology of C_c-vague convergence. Hence, we first have to do some surgery.
Let $(X^{n_l,m})_{l\in\mathbb{N}}$ be some subsequence of $(X^{n,m})_{n\in\mathbb{N}}$. Assume that $X^{n_l,m}_{(-L-1,L+1)^2}$ is a solution to the martingale problem for $(L_{n,m}, D(L_{n,m}))$ on $(-L-1, L+1)^2$ which is coupled to $X^{n_l,m}_{(0,L)^2} := X^{n_l,m}$ on $(0,L)^2$ such that $X^{n_l,m}_{(0,L)^2} \le X^{n_l,m}_{(-L-1,L+1)^2}$ in the sense of measures. This can be achieved by first defining the process on $(-L-1, L+1)^2$ and then killing all particles that leave $(0,L)^2$. Both processes start in $\mu_{n,m}$ after extending the measure by 0.
By using the relative compactness, we can find a subsubsequence $(n_l^*)_{l\in\mathbb{N}}$ such that both processes converge weakly. Let $Z^{l,m}_{(0,L)^2}, Z^{l,m}_{(-L-1,L+1)^2}, Z^m_{(0,L)^2}$ and $Z^m_{(-L-1,L+1)^2}$ be their almost surely converging counterparts from the Skorokhod representation theorem. We claim that for any $\varphi \in C_c((-L-1, L+1)^2)$ such that $\mathbb{1}_{(0,L)^2} \le \varphi$, $\langle \mathbb{1}_{(0,L)^2}, Z^m_{(0,L)^2}(t)\rangle \le \langle \varphi, Z^m_{(-L-1,L+1)^2}(t)\rangle$.
Proof of the claim: We will see in Theorem 2.42 below an argument which shows that $Z^m_{(0,L)^2}$ is a.s. continuous. Hence it follows from Theorem 2.28, that a.s. for any $0 \le t \le T$, $Z^{l,m}_{(0,L)^2}(t) \to Z^m_{(0,L)^2}(t)$, C_c-vaguely. Since $\mathrm{Law}((X^{n_l^*,m}_{(0,L)^2}, X^{n_l^*,m}_{(-L-1,L+1)^2}))$ is concentrated on sets, where $X^{n_l^*,m}_{(0,L)^2} \le X^{n_l^*,m}_{(-L-1,L+1)^2}$ and the laws coincide, the same

follows for the counterparts. By [Kle14, Lemma 13.15],

$$\begin{aligned}\langle \mathbb{1}_{(0,L)^2}, Z^m_{(0,L)^2}(t)\rangle &\leq \liminf_{l\to\infty}\langle \mathbb{1}_{(0,L)^2}, Z^{l,m}_{(0,L)^2}(t)\rangle \\ &\leq \liminf_{l\to\infty}\langle \varphi, Z^{l,m}_{(-L-1,L+1)^2}(t)\rangle = \langle \varphi, Z^m_{(-L-1,L+1)^2}(t)\rangle.\end{aligned}$$

This yields the claim.
Let $0 < t < T$ and let λ_l be is as in Lemma 2.27. Then, for $\varphi \in C_c((-L-1, L+1)^2)$, $\varphi \geq 0$, such that $\mathbb{1}_{(0,L)^2} \leq \varphi$,

$$\begin{aligned}\langle \mathbb{1}_{(0,L)^2}, Z^{l,m}_{(0,L)^2}(t)\rangle &\leq \sup_{0\leq s\leq \lambda_l(T)} \langle \mathbb{1}_{(0,L)^2}, Z^{l,m}_{(0,L)^2}(\lambda_l^{-1}(s))\rangle \\ &\leq \sup_{0\leq s\leq \lambda_l(T)} \langle \varphi, Z^{l,m}_{(-L-1,L+1)^2}(\lambda_l^{-1}(s))\rangle.\end{aligned}$$

We claim that as $l \to \infty$,

$$\sup_{0\leq s\leq \lambda_l(T)} |\langle \varphi, Z^{l,m}_{(-L-1,L+1)^2}(\lambda_l^{-1}(s))\rangle - \langle \varphi, Z^m_{(-L-1,L+1)^2}(s)\rangle| \to 0.$$

Proof of the claim: Assume not, then there exists some $\varepsilon > 0$ and a sequence $(s_l)_{l\in\mathbb{N}}$, $0 \leq s_l \leq \lambda_l(T)$, such that for any $l \in \mathbb{N}$,

$$|\langle \varphi, Z^{l,m}_{(-L-1,L+1)^2}(\lambda_l^{-1}(s_l))\rangle - \langle \varphi, Z^m_{(-L-1,L+1)^2}(s_l)\rangle| \geq \varepsilon.$$

We may choose $\varphi = f_{k^*}$, $k^* \in \mathbb{N}$, for some f_{k^*} as in Theorem 2.25. However, then

$$\begin{aligned}&\sup_{0\leq s\leq \lambda_l(T)} \sum_{k=1}^{\infty} 2^{-k}(1 - \exp(-|\langle f_k, Z^{l,m}_{(-L-1,L+1)^2}(\lambda_l^{-1}(s))\rangle - \langle f_k, Z^m_{(-L-1,L+1)^2}(s)\rangle|)) \\ &\geq 2^{-k^*}(1 - \exp(-\varepsilon)),\end{aligned}$$

in contradiction to Lemma 2.27. This yields the claim.
We get

$$\begin{aligned}\langle \mathbb{1}_{(0,L)^2}, Z^{l,m}_{(0,L)^2}(t)\rangle \leq &\sup_{0\leq s\leq \lambda_l(T)} |\langle \varphi, Z^{l,m}_{(-L-1,L+1)^2}(\lambda_l^{-1}(s))\rangle - \langle \varphi, Z^m_{(-L-1,L+1)^2}(s)\rangle| \\ &+ \sup_{0\leq s\leq \lambda_l(T)} |\langle \varphi, Z^m_{(-L-1,L+1)^2}(s)\rangle|.\end{aligned}$$

Since we may assume that for l sufficiently large, $\lambda_l(T) \leq T + 1$, it follows that $\langle \mathbb{1}_{(0,L)^2}, Z^{l,m}_{(0,L)^2}(t)\rangle$ is uniformly bounded in l and $0 \leq t \leq T$ by the claim above and the fact that $[0, T+1] \ni t \mapsto Z^m_{(-L-1,L+1)^2}(t)$ maps into the space of locally finite measures.
Consequently, by Theorem 2.28 and Lemma 2.36 we may assume that a.s. for any $0 \leq t \leq T$, $Z^{l,m}_{(0,L)^2}(t) \to Z^m_{(0,L)^2}(t)$, C_0-vaguely.

Let $\phi \in \mathrm{Dom}(\mathscr{H}_{\boldsymbol{\xi}_m})$, $\phi \geq 0$. We first show condition (17). We get that with the notation of Theorem 2.22, pointwise by the dominated convergence theorem,

$$\int_t^{t+s} \left\langle -\mathscr{H}_{\boldsymbol{\xi}_m}\phi + \frac{1}{2}\phi^2, Z^{l,m}_{(0,L)^2}(u) \right\rangle \exp(-\langle \phi, Z^{l,m}_{(0,L)^2}(u)\rangle) du \prod_{i=1}^k h_i(Z^{l,m}_{(0,L)^2}(t_i))$$
$$\to \int_t^{t+s} \left\langle -\mathscr{H}_{\boldsymbol{\xi}_m}\phi + \frac{1}{2}\phi^2, Z^{m}_{(0,L)^2}(u) \right\rangle \exp(-\langle \phi, Z^{m}_{(0,L)^2}(u)\rangle) du \prod_{i=1}^k h_i(Z^{m}_{(0,L)^2}(t_i)),$$

where we used the convergence and a.s. uniform boundedness of $\langle \mathbb{1}_{(0,L)^2}, Z^{l,m}_{(0,L)^2}(u)\rangle$ in l and $0 < u \leq t+s$. What is more, pointwise

$$\exp(-\langle \phi, Z^{l,m}_{(0,L)^2}(t)\rangle) \prod_{i=1}^k h_i(Z^{l,m}_{(0,L)^2}(t_i)) \to \exp(-\langle \phi, Z^{m}_{(0,L)^2}(t)\rangle) \prod_{i=1}^k h_i(Z^{m}_{(0,L)^2}(t_i)).$$

Since from here on, all arguments shall involve (probabilistic) distributions, we switch back to the original processes, using that they have the same law as their Skorokhod counterparts. It suffices to establish uniform integrability to show (17). However,

$$\mathbb{E}\left(\left(\int_t^{t+s} \left\langle -\mathscr{H}_{\boldsymbol{\xi}_m}\phi + \frac{1}{2}\phi^2, X^{n_l^*,m}_u \right\rangle \exp(-\langle \phi, X^{n_l^*,m}_u\rangle) du \prod_{i=1}^k h_i(X^{n_l^*,m}_{t_i})\right)^2\right)$$
$$\lesssim s^2 \left\| -\mathscr{H}_{\boldsymbol{\xi}_m}\phi + \frac{1}{2}\phi^2 \right\|^2_{L^\infty} \sup_{0 \leq u \leq t+s} \sup_{n \geq N_0} \mathbb{E}(\langle \mathbb{1}_{(0,L)^2}, X^{n,m}_u\rangle^2).$$

In order to establish (16), let us note: The integrability of the limit $Z^m_{(0,L)^2}$ follows by an application of Fatou's lemma and the comparison argument above, see Lemma 2.38 for the details.

For condition (18), let $f_n := 1 - \phi/n$. We can choose n large enough such that $f_n > 0$ and $\exp(\langle \log(f_n), \cdot\rangle) \in D(L^y_{n,m})$. We get by the martingale problem for $(L^y_{n,m}, D(L^y_{n,m}))$ that

$$E^{n\log(f_n)}_{n,m}(t) := \exp(\langle \log(f_n), nX^{n,m}_t\rangle) - \exp(\langle \log(f_n), nX^{n,m}_0\rangle)$$
$$- \int_0^{nt} \left\langle \frac{G^{\mathfrak{a}}_n f_n + \Phi_{n,m}(f_n) - f_n}{f_n}, nX^{n,m}_{s/n} \right\rangle \exp\left(\left\langle \log(f_n), nX^{n,m}_{s/n}\right\rangle\right) ds$$

is a martingale. We have $\Phi_{n,m}(f_n) = \Phi_{n,m}(1) - \phi n^{-1}\Phi'_{n,m}(1) + 1/2\phi^2 n^{-2}\Phi''_{n,m}(1)$. Hence we get

$$E^{n\log(f_n)}_{n,m}(t) = \exp(\langle n\log(f_n), X^{n,m}_t\rangle) - \exp(\langle n\log(f_n), X^{n,m}_0\rangle)$$
$$- \int_0^t \left\langle \frac{n\Delta f_n + n\phi(1 - \Phi'_{n,m}(1)) + 1/2\phi^2\Phi''_{n,m}(1) + n^2(\Phi_{n,m}(1) - 1)}{f_n}, X^{n,m}_s \right\rangle$$
$$\times \exp(\langle n\log(f_n), X^{n,m}_s\rangle) ds.$$

By the specific form of the probability generating function, we have $\Phi_{n,m}(1) = 1$, $n(1 - \Phi'_{n,m}(1)) = n(1 - 2\zeta^{+1}_{n,m}) = -(\xi_m - c_m)$ and $\Phi''_{n,m}(1) = 2\zeta^{+1}_{n,m}$. So,

$$\begin{aligned}E^{n\log(f_n)}_{n,m}(t) =& \exp(\langle n\log(f_n), X^{n,m}_t\rangle) - \exp(\langle n\log(f_n), X^{n,m}_0\rangle)\\ &- \int_0^t \left\langle \frac{-\mathscr{H}_{\boldsymbol{\xi}_m}\phi + 1/2\phi^2 2\zeta^{+1}_{n,m}}{f_n}, X^{n,m}_s \right\rangle \exp(\langle n\log(f_n), X^{n,m}_s\rangle) ds.\end{aligned}$$

Let $h(\cdot) = \exp(-\langle \phi, \cdot\rangle)$ for $\phi \in \mathrm{Dom}(\mathscr{H}_{\boldsymbol{\xi}_m})$, $\phi \geq 0$. Let $k \in \mathbb{N}$, $(h_i)_{i\in\{1,\dots,k\}}$ be such that $h_i \in C_b(M((0,L)^2))$ and $0 \leq t_1 < \cdots < t_k \leq t < t + s$. We compute

$$\begin{aligned}&\mathbb{E}\left(\left(h(X^{n,m}_{t+s}) - h(X^{n,m}_t) - \int_t^{t+s} L_m h(X^{n,m}_u) du\right)\prod_{i=1}^k h_i(X^{n,m}_{t_i})\right)\\ &= \mathbb{E}\Bigg(\Bigg(\exp(-\langle \phi, X^{n,m}_{t+s}\rangle) - \exp(\langle n\log(f_n), X^{n,m}_{t+s}\rangle)\\ &- \exp(-\langle \phi, X^{n,m}_t\rangle) + \exp(\langle n\log(f_n), X^{n,m}_t\rangle)\\ &+ \int_t^{t+s} \left\langle \frac{-\mathscr{H}_{\boldsymbol{\xi}_m}\phi + 1/2\phi^2 2\zeta^{+1}_{n,m}}{f_n}, X^{n,m}_u \right\rangle \exp(\langle n\log(f_n), X^{n,m}_u\rangle)\\ &- \left\langle -\mathscr{H}_{\boldsymbol{\xi}_m}\phi + 1/2\phi^2, X^{n,m}_u \right\rangle \exp(-\langle \phi, X^{n,m}_u\rangle) du\Bigg)\prod_{i=1}^k h_i(X^{n,m}_{t_i})\Bigg).\end{aligned} \tag{23}$$

We get by Lemma 2.35 that $n\log(f_n) = \log((1-\phi/n)^n) \to -\phi$ uniformly in $[0,L]^2$. Also, $\phi \geq 0$ and $X^{n,m}$ is a positive measure. We get for $K > 0$,

$$\begin{aligned}&\mathbb{E}\left((\exp(-\langle \phi, X^{n,m}_t\rangle) - \exp(\langle n\log(f_n), X^{n,m}_t\rangle))\prod_{i=1}^k h_i(X^{n,m}_{t_i})\right)\\ &= \mathbb{E}\left((\exp(-\langle \phi, X^{n,m}_t\rangle) - \exp(\langle n\log(f_n), X^{n,m}_t\rangle))\prod_{i=1}^k h_i(X^{n,m}_{t_i}); \left\langle \mathbb{1}_{(0,L)^2}, X^{n,m}_t\right\rangle > K\right)\\ &+ \mathbb{E}\left((\exp(-\langle \phi, X^{n,m}_t\rangle) - \exp(\langle n\log(f_n), X^{n,m}_t\rangle))\prod_{i=1}^k h_i(X^{n,m}_{t_i}); \left\langle \mathbb{1}_{(0,L)^2}, X^{n,m}_t\right\rangle \leq K\right).\end{aligned}$$

Note that $n\log(f_n(x)) = n\log(1 - \phi(x)/n) \leq 0$ as we have chosen n large enough such that $0 < 1 - \phi(x)/n < 1$ by the non-negativity of ϕ. By the non-positivity of the exponents for n large enough and the boundedness of the h_i,

$$\begin{aligned}&\mathbb{E}\left((\exp(-\langle \phi, X^{n,m}_t\rangle) - \exp(\langle n\log(f_n), X^{n,m}_t\rangle))\prod_{i=1}^k h_i(X^{n,m}_{t_i}); \left\langle \mathbb{1}_{(0,L)^2}, X^{n,m}_t\right\rangle > K\right)\\ &\lesssim \mathbb{P}(\langle \mathbb{1}_{(0,L)^2}, X^{n,m}_t\rangle > K) \leq 1/K \sup_{n\geq N_0} \mathbb{E}(\langle \mathbb{1}_{(0,L)^2}, X^{n,m}_t\rangle),\end{aligned}$$

which vanishes as $K \to \infty$ uniformly in n by Lemma 2.18. Inside the second expectation, where $\langle \mathbb{1}_{(0,L)^2}, X_t^{n,m} \rangle \leq K$,

$$\langle |-\phi - n\log(f_n)|, X_t^{n,m} \rangle \leq \langle \mathbb{1}_{(0,L)^2}, X_t^{n,m} \rangle \| -\phi - n\log(f_n) \|_{L^\infty} \leq K \| -\phi - n\log(f_n) \|_{L^\infty}. \tag{24}$$

By using Taylor's theorem

$$\begin{aligned}
&\exp(-\langle \phi, X_t^{n,m} \rangle) - \exp(\langle n\log(f_n), X_t^{n,m} \rangle) \\
&= \exp(\langle n\log(f_n), X_t^{n,m} \rangle)(-\langle \phi, X_t^{n,m} \rangle - \langle n\log(f_n), X_t^{n,m} \rangle) \\
&+ o(-\langle \phi, X_t^{n,m} \rangle - \langle n\log(f_n), X_t^{n,m} \rangle).
\end{aligned} \tag{25}$$

We conclude that the term

$$\mathbb{E}\left((\exp(-\langle \phi, X_t^{n,m} \rangle) - \exp(\langle n\log(f_n), X_t^{n,m} \rangle)) \prod_{i=1}^k h_i(X_{t_i}^{n,m}); \langle \mathbb{1}_{(0,L)^2}, X_t^{n,m} \rangle \leq K \right)$$

vanishes as well. This implies that the first differences in (23) vanish.
Furthermore, $2\zeta_{n,m}^{+1} \to 1$ and $f_n \to 1$ as $n \to \infty$ uniformly in $[0,L]^2$. For the integral in (23) we get

$$\begin{aligned}
&\int_t^{t+s} \left(\left\langle \frac{-\mathscr{H}_{\boldsymbol{\xi}_m}\phi + 1/2\phi^2 2\zeta_{n,m}^{+1}}{f_n}, X_u^{n,m} \right\rangle - \left\langle -\mathscr{H}_{\boldsymbol{\xi}_m}\phi + \frac{1}{2}\phi^2, X_u^{n,m} \right\rangle \right) \\
&\times \exp(\langle n\log(f_n), X_u^{n,m} \rangle) \\
&- \left\langle -\mathscr{H}_{\boldsymbol{\xi}_m}\phi + \frac{1}{2}\phi^2, X_u^{n,m} \right\rangle (\exp(-\langle \phi, X_u^{n,m} \rangle) - \exp(\langle n\log(f_n), X_u^{n,m} \rangle)) du.
\end{aligned}$$

For the first term by the non-positivity of the exponent for n large enough,

$$\begin{aligned}
&\mathbb{E}\Bigg(\Bigg(\int_t^{t+s} \left(\left\langle \frac{-\mathscr{H}_{\boldsymbol{\xi}_m}\phi + 1/2\phi^2 2\zeta_{n,m}^{+1}}{f_n}, X_u^{n,m} \right\rangle - \left\langle -\mathscr{H}_{\boldsymbol{\xi}_m}\phi + \frac{1}{2}\phi^2, X_u^{n,m} \right\rangle \right) \\
&\times \exp(\langle n\log(f_n), X_u^{n,m} \rangle) du \Bigg) \prod_{i=1}^k h_i(X_{t_i}^{n,m}) \Bigg) \\
&\lesssim \left\| \frac{-\mathscr{H}_{\boldsymbol{\xi}_m}\phi + 1/2\phi^2 2\zeta_{n,m}^{+1}}{f_n} - \left(-\mathscr{H}_{\boldsymbol{\xi}_m}\phi + \frac{1}{2}\phi^2 \right) \right\|_{L^\infty} \int_t^{t+s} \mathbb{E}\left(\langle \mathbb{1}_{(0,L)^2}, X_u^{n,m} \rangle \right) du \to 0,
\end{aligned}$$

where we used that $f_n \to 1$ uniformly, along with the other terms, and that the first moment of $X_t^{n,m}$ is integrable over finite time horizons uniformly in $n \geq N_0$. To be more precise, we note that

$$\begin{aligned}
&\sup_{x \in [0,L]^2} \left| \frac{-\mathscr{H}_{\boldsymbol{\xi}_m}\phi(x) + 1/2\phi^2(x) 2\zeta_{n,m}^{+1}(x)}{f_n(x)} - \left(-\mathscr{H}_{\boldsymbol{\xi}_m}\phi(x) + 1/2\phi^2(x) \right) \right| \\
&\leq \frac{\sup_{x \in [0,L]^2} |1/(2n)\phi^2(x)(\xi_m(x) - c_m) - ((\Delta + \xi_m(x) - c_m)\phi(x))\phi(x)/n + 1/(2n)\phi^3(x)|}{\inf_{x \in [0,L]^2} |1 - \phi(x)/n|} \\
&\to 0,
\end{aligned}$$

as $n \to \infty$. For the second term by Fubini's theorem and Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}\Bigg(\int_t^{t+s} \left\langle -\mathscr{H}_{\boldsymbol{\xi}_m}\phi + \frac{1}{2}\phi^2, X_u^{n,m} \right\rangle (\exp(-\langle \phi, X_u^{n,m}\rangle) - \exp(\langle n\log(f_n), X_u^{n,m}\rangle))du \\
&\times \prod_{i=1}^k h_i(X_{t_i}^{n,m}) \Bigg) \\
&\lesssim \int_t^{t+s} \mathbb{E}\left(\left\langle -\mathscr{H}_{\boldsymbol{\xi}_m}\phi + \frac{1}{2}\phi^2, X_u^{n,m} \right\rangle^2 \right)^{1/2} \\
&\times \mathbb{E}\left((\exp(-\langle \phi, X_u^{n,m}\rangle) - \exp(\langle n\log(f_n), X_u^{n,m}\rangle))^2 \right)^{1/2} du \\
&\le \left(\int_t^{t+s} \mathbb{E}\left(\left\langle -\mathscr{H}_{\boldsymbol{\xi}_m}\phi + \frac{1}{2}\phi^2, X_u^{n,m} \right\rangle^2 \right) du \right)^{1/2} \\
&\times \left(\int_t^{t+s} \mathbb{E}\left((\exp(-\langle \phi, X_u^{n,m}\rangle) - \exp(\langle n\log(f_n), X_u^{n,m}\rangle))^2 \right) du \right)^{1/2}
\end{aligned}$$

We get similarly to the above,

$$\begin{aligned}
&\mathbb{E}\left((\exp(-\langle \phi, X_u^{n,m}\rangle) - \exp(\langle n\log(f_n), X_u^{n,m}\rangle))^2 \right) \\
&\le \mathbb{E}\left((\exp(-\langle \phi, X_u^{n,m}\rangle) - \exp(\langle n\log(f_n), X_u^{n,m}\rangle))^2; \langle \mathbb{1}_{(0,L)^2}, X_u^{n,m}\rangle > K \right) \\
&+ \mathbb{E}\left((\exp(-\langle \phi, X_u^{n,m}\rangle) - \exp(\langle n\log(f_n), X_u^{n,m}\rangle))^2; \langle \mathbb{1}_{(0,L)^2}, X_u^{n,m}\rangle \le K \right).
\end{aligned}$$

So,

$$\begin{aligned}
&\mathbb{E}\left((\exp(-\langle \phi, X_u^{n,m}\rangle) - \exp(\langle n\log(f_n), X_u^{n,m}\rangle))^2; \langle \mathbb{1}_{(0,L)^2}, X_u^{n,m}\rangle > K \right) \\
&\le 4\mathbb{P}(\langle \mathbb{1}_{(0,L)^2}, X_u^{n,m}\rangle > K) \le 4/K \sup_{n \ge N_0} \mathbb{E}(\langle \mathbb{1}_{(0,L)^2}, X_u^{n,m}\rangle),
\end{aligned}$$

and the RHS is integrable over finite time horizons uniformly over $n \ge N_0$ and vanishes as $K \to \infty$. Next,

$$\mathbb{E}\left((\exp(-\langle \phi, X_u^{n,m}\rangle) - \exp(\langle n\log(f_n), X_u^{n,m}\rangle))^2; \langle \mathbb{1}_{(0,L)^2}, X_u^{n,m}\rangle \le K \right) \to 0,$$

as $n \to \infty$ using (24) and (25). Finally,

$$\mathbb{E}\left(\left\langle -\mathscr{H}_{\boldsymbol{\xi}_m}\phi + \frac{1}{2}\phi^2, X_u^{n,m} \right\rangle^2 \right) \le \left\| -\mathscr{H}_{\boldsymbol{\xi}_m}\phi + \frac{1}{2}\phi^2 \right\|_{L^\infty}^2 \mathbb{E}\left(\left\langle \mathbb{1}_{(0,L)^2}, X_u^{n,m} \right\rangle^2 \right),$$

and by Lemma 2.20, this term is integrable over finite time horizons uniformly in $n \ge N_0$. □

Having established tightness and the convergence of generators, we now turn towards uniqueness of the limiting martingale problem. Note that we already have the weak convergence of a subsequence $X^{n,m} \to X^m$ as $n \to \infty$, see the proof of Theorem 2.22.

Assumption: In the following, if we denote the starting point of X^m by $\mu_m \in M_F((0,L)^2)$, we implicitly assume that there is some sequence $n\mu_{n,m} \in E$ such that $\mu_{n,m} \to \mu_m$ in $M((0,L)^2)$ and $\sup_{n \geq N_0} \langle \mathbb{1}_{(0,L)^2}, \mu_{n,m} \rangle < \infty$. Also, we denote by $(\mathcal{F}_t^m)_{t \geq 0}$ the natural filtration of X^m.

Uniqueness can be achieved via the injectivity of Laplace transforms. In order to establish an exponential martingale for X^m, a solution to the martingale problem for $(L_m, D(L_m))$, we first derive some preliminary moment bounds.

Lemma 2.38
Let X^m be a solution to the martingale problem for $(L_m, D(L_m))$ started in some $\mu_m \in M_F((0,L)^2)$. It holds for $T > 0$, $m \in \mathbb{N}$, that

$$\sup_{0 \leq t \leq T} \mathbb{E}\left(\left\langle \mathbb{1}_{(0,L)^2}, X_t^m \right\rangle^4 \right) < \infty.$$

In particular by Hölder's inequality the same holds true for lower order moments as well. Also, X^m is actually a finite measure a.s..

Proof
By the Skorokhod representation theorem, we may assume that $X^{n,m} \to X^m$ a.s., at least for a subsequence. Hence we can apply Lemma 2.28, Fatou's lemma and Lemma 2.20 to get for $\phi \in C_c((0,L)^2)$,

$$\mathbb{E}\left(\langle \phi, X_t^m \rangle^4 \right) \leq \liminf_{n \to \infty} \mathbb{E}\left(\langle \phi, X_t^{n,m} \rangle^4 \right) \leq \sup_{n \geq N_0} \mathbb{E}\left(\langle \phi, X_t^{n,m} \rangle^4 \right),$$

which is uniformly bounded over finite time horizons for N_0 sufficiently large.
We cannot choose $\phi = \mathbb{1}_{(0,L)^2}$ in the above, since we equipped $M((0,L)^2)$ with the vague topology in duality with $C_c((0,L)^2)$- functions. In order to show that our killed mollified Super Brownian Motion does not explode towards the boundary, we use again a comparison argument.
Let $X^{n,m}_{(0,L)^2}, X^{n,m}_{(-L-1,L+1)^2}$ be solutions to the martingale problem for $(L_{n,m}, D(L_{n,m}))$ on $(0,L)^2$ and $(-L-1, L+1)^2$ respectively, both started in the same $\mu_{n,m}$, $n\mu_{n,m} \in E$, such that $\mu_{n,m} \to \mu_m \in M((0,L)^2)$ and $\sup_{n \geq N_0} \langle \mathbb{1}_{(0,L)^2}, \mu_{n,m} \rangle < \infty$. We assume as in the proof of Theorem 2.37, that we can define both processes on the same underlying probability space and that $X^{n,m}_{(0,L)^2}(t) \leq X^{n,m}_{(-L-1,L+1)^2}(t) \restriction_{(0,L)^2}$ holds for any $t \geq 0$. We get $(X^{n,m}_{(0,L)^2}, X^{n,m}_{(-L-1,L+1)^2}) \to (X^m_{(0,L)^2}, X^m_{(-L-1,L+2)^2})$ weakly with $X^m_{(0,L)^2}, X^m_{(-L-1,L+1)^2}$ being killed mollified Super Brownian Motions on $(0,L)^2$ and $(-L-1, L+1)^2$ respectively. By the Skorokhod representation theorem, there are some $(Z^{n,m}_{(0,L)^2}, Z^{n,m}_{(-L-1,L+1)^2}) \to (Z^m_{(0,L)^2}, Z^m_{(-L-1,L+1)^2})$ with $\mathrm{Law}((Z^{n,m}_{(0,L)^2}, Z^{n,m}_{(-L-1,L+1)^2})) = \mathrm{Law}((X^{n,m}_{(0,L)^2}, X^{n,m}_{(-L-1,L+1)^2}))$ for any $n \geq N_0$ and $\mathrm{Law}((Z^m_{(0,L)^2}, Z^m_{(-L-1,L+1)^2})) = \mathrm{Law}((X^m_{(0,L)^2}, X^m_{(-L-1,L+1)^2}))$.

It follows as in the proof of Theorem 2.37, that for any $\phi \in C_c((-L-1,L+1)^2)$, $\mathbb{1}_{(0,L)^2} \leq \phi$, $\langle \mathbb{1}_{(0,L)^2}, Z^m_{(0,L)^2}(t)\rangle \leq \langle \phi, Z^m_{(-L-1,L+1)^2}(t)\rangle$.
Let $\phi \in C_c((-L-1,L+1)^2)$, $\phi \geq 0$, be such that $\mathbb{1}_{(0,L)^2} \leq \phi$. Then,

$$\begin{aligned}\mathbb{E}\left(\left\langle \mathbb{1}_{(0,L)^2}, X^m_{(0,L)^2}(t)\right\rangle^4\right) &= \mathbb{E}\left(\left\langle \mathbb{1}_{(0,L)^2}, Z^m_{(0,L)^2}(t)\right\rangle^4\right) \\ &\leq \mathbb{E}\left(\left\langle \phi, Z^m_{(-L-1,L+1)^2}(t)\right\rangle^4\right) = \mathbb{E}\left(\left\langle \phi, X^m_{(-L-1,L+1)^2}(t)\right\rangle^4\right)\end{aligned}$$

and the claim follows by the above. Finally, the finiteness of $Z^m_{(0,L)^2}$ follows by the local finiteness of $Z^m_{(-L-1,L+1)^2}$.

□

Since $\|\xi_m\|_{L^\infty} \to \infty$ as $m \to \infty$, such pointwise considerations do not yield a bound which is uniform in m, hence the terminology 'preliminary'. However, the above can still be used to extract some information about X^m.

Lemma 2.39
Let X^m be a solution to the martingale problem for $(L_m, D(L_m))$. Assume that $\phi \in C^1([0,\infty), C_0((0,L)^2))$, $\phi(t) \in Dom(\mathscr{H}_{\xi_m})$ for any $t \geq 0$ and $\phi \geq 0$. Also assume that $\mathscr{H}_{\xi_m}\phi \in C([0,\infty), C_0((0,L)^2))$. Then for $t \geq 0$,

$$\begin{aligned}\tilde{E}^\phi_m(t) &:= \exp(-\langle \phi(t), X^m_t\rangle) - \exp(-\langle \phi(0), X^m_0\rangle) \\ &\quad - \int_0^t \left\langle -\mathscr{H}_{\xi_m}\phi(s) + \frac{1}{2}\phi^2(s) - \partial_s\phi(s), X^m_s\right\rangle \exp(-\langle \phi(s), X^m_s\rangle) ds\end{aligned}$$

is a martingale.

Proof
We define for $\phi \in \mathrm{Dom}(\mathscr{H}_{\xi_m})$, $\phi \geq 0$, E^ϕ_m to be the martingale associated to the martingale problem for $(L_m, D(L_m))$. Let $(\pi_k)_{k\in\mathbb{N}}$ be a sequence of partitions of $[0,t]$ such that $\pi_k = (t^k_0, \ldots, t^k_k)$, $t^k_0 < \cdots < t^k_k$, $t^k_0 = 0$, $t^k_k = t$ and $\lim_{k\to\infty} \sup_{i\in\{1,\ldots,k\}} t^k_i - t^k_{i-1} = 0$. We compute now for ϕ as in the claim,

$$\begin{aligned}&\exp\left(-\langle \phi(t), X^m_t\rangle\right) - \exp\left(-\langle \phi(0), X^m_0\rangle\right) \\ &= \sum_{i=1}^k \exp\left(-\left\langle \phi(t^k_i), X^m_{t^k_i}\right\rangle\right) - \exp\left(-\left\langle \phi(t^k_i), X^m_{t^k_{i-1}}\right\rangle\right) \\ &+ \exp\left(-\left\langle \phi(t^k_i), X^m_{t^k_{i-1}}\right\rangle\right) - \exp\left(-\left\langle \phi(t^k_{i-1}), X^m_{t^k_{i-1}}\right\rangle\right).\end{aligned}$$

Consequently, it follows that

$$
\begin{aligned}
&\exp\left(-\langle \phi(t), X_t^m\rangle\right) - \exp\left(-\langle \phi(0), X_0^m\rangle\right) \\
&= \sum_{i=1}^{k} E_m^{\phi(t_i^k)}(t_i^k) - E_m^{\phi(t_i^k)}(t_{i-1}^k) \\
&+ \int_{t_{i-1}^k}^{t_i^k} \left\langle -\mathscr{H}_{\boldsymbol{\xi}_m}\phi(t_i^k) + \frac{1}{2}\phi(t_i^k)^2, X_s^m \right\rangle \exp\left(-\left\langle \phi(t_i^k), X_s^m\right\rangle\right) ds \\
&+ \frac{\exp\left(-\left\langle \phi(t_i^k), X_{t_{i-1}^k}^m\right\rangle\right) - \exp\left(-\left\langle \phi(t_{i-1}^k), X_{t_{i-1}^k}^m\right\rangle\right)}{t_i^k - t_{i-1}^k}(t_i^k - t_{i-1}^k).
\end{aligned}
$$

By the mean-value theorem there exists some $t_{i,*}^k \in (t_{i-1}^k, t_i^k)$ such that

$$
\begin{aligned}
&\frac{\exp\left(-\left\langle \phi(t_i^k), X_{t_{i-1}^k}^m\right\rangle\right) - \exp\left(-\left\langle \phi(t_{i-1}^k), X_{t_{i-1}^k}^m\right\rangle\right)}{t_i^k - t_{i-1}^k} \\
&= -\left\langle \partial_t\phi(t_{i,*}^k), X_{t_{i-1}^k}^m\right\rangle \exp\left(-\left\langle \phi(t_{i,*}^k), X_{t_{i-1}^k}^m\right\rangle\right).
\end{aligned}
$$

We define

$$
\begin{aligned}
\phi_k(t) &:= \sum_{i=1}^{k} \phi(t_i^k)\mathbb{1}_{[t_{i-1}^k, t_i^k)}(t), \quad \phi_{k,*}(t) := \sum_{i=1}^{k} \phi(t_{i,*}^k)\mathbb{1}_{[t_{i-1}^k, t_i^k)}(t), \\
\phi_k'(t) &:= \sum_{i=1}^{k} \partial_t\phi(t_{i,*}^k)\mathbb{1}_{[t_{i-1}^k, t_i^k)}(t), \quad X_k^m(t) := \sum_{i=1}^{k} X_{t_{i-1}^k}^m \mathbb{1}_{[t_{i-1}^k, t_i^k)}(t)
\end{aligned}
$$

and write

$$
\begin{aligned}
&\sum_{i=1}^{k} \int_{t_{i-1}^k}^{t_i^k} \left\langle -\mathscr{H}_{\boldsymbol{\xi}_m}\phi(t_i^k) + \frac{1}{2}\phi(t_i^k)^2, X_s^m \right\rangle \exp\left(-\left\langle \phi(t_i^k), X_s^m\right\rangle\right) ds \\
&= \int_0^t \left\langle -\mathscr{H}_{\boldsymbol{\xi}_m}\phi_k(s) + \frac{1}{2}\phi_k(s)^2, X_s^m \right\rangle \exp\left(-\langle \phi_k(s), X_s^m\rangle\right) ds.
\end{aligned}
$$

Since $\phi_k \to \phi$ and $\mathscr{H}_{\boldsymbol{\xi}_m}\phi_k \to \mathscr{H}_{\boldsymbol{\xi}_m}\phi$ pointwise and uniformly bounded, we get by the dominated convergence theorem that for each realisation of X^m,

$$
\begin{aligned}
&\sum_{i=1}^{k} \int_{t_{i-1}^k}^{t_i^k} \left\langle -\mathscr{H}_{\boldsymbol{\xi}_m}\phi(t_i^k) + \frac{1}{2}\phi(t_i^k)^2, X_s^m \right\rangle \exp\left(-\left\langle \phi(t_i^k), X_s^m\right\rangle\right) ds \\
&\to \int_0^t \left\langle -\mathscr{H}_{\boldsymbol{\xi}_m}\phi(s) + \frac{1}{2}\phi(s)^2, X_s^m \right\rangle \exp\left(-\langle \phi(s), X_s^m\rangle\right) ds.
\end{aligned}
$$

Similarly,

$$\sum_{i=1}^{k} \frac{\exp\left(-\left\langle \phi(t_i^k), X^m_{t_{i-1}^k}\right\rangle\right) - \exp\left(-\left\langle \phi(t_{i-1}^k), X^m_{t_{i-1}^k}\right\rangle\right)}{t_i^k - t_{i-1}^k}(t_i^k - t_{i-1}^k)$$
$$= \int_0^t -\langle \phi_k'(s), X_k^m(s)\rangle \exp\left(-\langle \phi_{k,*}(s), X_k^m(s)\rangle\right) ds.$$

It holds that $\phi_k' \to \partial_t \phi$, $\phi_{k,*} \to \phi$ pointwise and uniformly bounded. Also $X_k^m \to X^m$, C_0-vaguely as $k \to \infty$ pointwise in time and uniformly bounded, see also Lemma 2.43 below. It follows by the dominated convergence theorem, that realisation-wise,

$$\sum_{i=1}^{k} \frac{\exp\left(-\left\langle \phi(t_i^k), X^m_{t_{i-1}^k}\right\rangle\right) - \exp\left(-\left\langle \phi(t_{i-1}^k), X^m_{t_{i-1}^k}\right\rangle\right)}{t_i^k - t_{i-1}^k}(t_i^k - t_{i-1}^k)$$
$$\to \int_0^t -\langle \partial_s \phi(s), X_s^m\rangle \exp\left(-\langle \phi(s), X_s^m\rangle\right) ds.$$

In order to establish L^1-convergence, we verify uniform integrability. We compute

$$\mathbb{E}\left(\left(\sum_{i=1}^{k}\int_{t_{i-1}^k}^{t_i^k} \left\langle -\mathscr{H}_{\boldsymbol{\xi}_m}\phi(t_i^k) + \frac{1}{2}\phi(t_i^k)^2, X_s^m\right\rangle \exp\left(-\left\langle \phi(t_i^k), X_s^m\right\rangle\right) ds\right)^2\right)$$
$$= \mathbb{E}\left(\left(\int_0^t \left\langle -\mathscr{H}_{\boldsymbol{\xi}_m}\phi_k(s) + \frac{1}{2}\phi_k(s)^2, X_s^m\right\rangle \exp\left(-\langle \phi_k(s), X_s^m\rangle\right) ds\right)^2\right)$$
$$\le \sup_{0\le s\le t}\left\| -\mathscr{H}_{\boldsymbol{\xi}_m}\phi(s) + \frac{1}{2}\phi^2(s)\right\|_{L^\infty}^2 t\int_0^t \mathbb{E}\left(\left\langle \mathbb{1}_{(0,L)^2}, X_s^m\right\rangle^2\right) ds < \infty.$$

The RHS is bounded by Lemma 2.38. For the second term,

$$\mathbb{E}\left(\left(\sum_{i=1}^{k} \frac{\exp\left(-\left\langle \phi(t_i^k), X^m_{t_{i-1}^k}\right\rangle\right) - \exp\left(-\left\langle \phi(t_{i-1}^k), X^m_{t_{i-1}^k}\right\rangle\right)}{t_i^k - t_{i-1}^k}(t_i^k - t_{i-1}^k)\right)^2\right)$$
$$= \mathbb{E}\left(\left(\int_0^t -\langle \phi_k'(s), X_k^m(s)\rangle \exp\left(-\langle \phi_{k,*}(s), X_k^m(s)\rangle\right) ds\right)^2\right)$$
$$\lesssim \sup_{0\le s\le t}\|\partial_t\phi\|_{L^\infty}^2 t^2 \sup_{0\le s\le t}\mathbb{E}(\langle \mathbb{1}_{(0,L)^2}, X_s^m\rangle^2) < \infty.$$

Let $\tilde{E}_m^\phi(t) := \lim_{k\to\infty}\sum_{i=1}^k E_m^{\phi(t_i^k)}(t_i^k) - E_m^{\phi(t_i^k)}(t_{i-1}^k)$. In order to prove the martingale property for $\tilde{E}_m^\phi(t)$, note that for $s < t$,

$$\mathbb{E}\left(\tilde{E}_m^\phi(t)|\mathcal{F}_s^m\right) = \lim_{k\to\infty}\mathbb{E}\left(\sum_{i=1}^k E_m^{\phi(t_i^k)}(t_i^k) - E_m^{\phi(t_i^k)}(t_{i-1}^k)\middle|\mathcal{F}_s^m\right)$$

and on the RHS we can choose a subsequence of partitions containing s as an element. Then the martingale property of the limit follows by the martingale property of the $E_m^{\phi(t_i^k)}$.

□

We can now choose a convenient time dependent function to derive an exponential martingale. This function is precisely the solution to the Evolution Equation for the killed mollified Super Brownian Motion discussed in Section 1.9.

Lemma 2.40
Let $\psi \in Dom(\mathscr{H}_{\mathfrak{\xi}_m}) \cap \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$, $\psi \geq 0$, $\psi \neq 0$, $t > 0$, and $\gamma > 0$ be sufficiently small. Then it holds that the process for $s \leq t$ given by

$$E_m^{\gamma\psi}(s,t) := \exp(-\langle U_{t-s}^m(\gamma\psi), X_s^m\rangle) - \exp(-\langle U_t^m(\gamma\psi), X_0^m\rangle)$$

is a bounded martingale.

Proof
The assertion is a consequence of Lemma 2.39 and Lemma 1.63. The non-negativity follows from Conjecture 1.65.

□

We can now finally prove the uniqueness of the killed mollified Super Brownian Motion using the (non-linear) semigroup $t \mapsto U_t^m$.

Lemma 2.41
Any solution to the martingale problem for $(L_m, D(L_m))$ started in $\mu_m \in M_F((0,L)^2)$ is unique.

Proof
Let $k \in \mathbb{N}$, $0 \leq t_1 \leq \cdots \leq t_k \leq t$ and $(\phi_i)_{i\in\{1,\ldots,k\}}$ be such that $\phi_i \in \mathrm{Dom}(\mathscr{H}_{\mathfrak{\xi}_m}) \cap \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$, $\phi_i \geq 0$, $\phi_i \neq 0$ for any $i \in \{1,\ldots,k\}$. Let $\gamma_i > 0$, $i \in \{1,\ldots,k\}$. Let X^m be any solution to the martingale problem $(L_m, D(L_m))$ started in μ_m. We compute with Lemma 2.40,

$$\begin{aligned}
&\mathbb{E}(\exp(-\langle\gamma_1\phi_1, X_{t_1}^m\rangle)\cdot\cdots\cdot\exp(-\langle\gamma_k\phi_k, X_{t_k}^m\rangle))\\
&=\mathbb{E}(\exp(-\langle\gamma_1\phi_1, X_{t_1}^m\rangle)\cdot\cdots\cdot\exp(-\langle\gamma_{k-1}\phi_{k-1}, X_{t_{k-1}}^m\rangle)\exp(-\langle U_{t_k-t_{k-1}}^m(\gamma_k\phi_k), X_{t_{k-1}}^m\rangle))\\
&=\mathbb{E}(\exp(-\langle\gamma_1\phi_1, X_{t_1}^m\rangle)\cdot\cdots\cdot\exp(-\langle\gamma_{k-1}\phi_{k-1} + U_{t_k-t_{k-1}}^m(\gamma_k\phi_k), X_{t_{k-1}}^m\rangle)).
\end{aligned}$$

We assumed that γ_k is sufficiently small. Using that $0 \leq U_{t_k-t_{k-1}}^m\phi_k \in \mathrm{Dom}(\mathscr{H}_{\mathfrak{\xi}_m}) \cap \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$, we can proceed inductively. Let us point out an important technical step: When iterating the Evolution Equation as above, one arrives at the expression $\gamma_{k-1}\phi_{k-1} + U_{t_k-t_{k-1}}^m(\gamma_k\phi_k)$ with $\gamma_{k-1}, \gamma_k > 0$. The constant $\gamma_k = \gamma_k(\phi_k, t_k - t_{k-1})$ was chosen sufficiently small for the construction of the Evolution Equation carried out in Lemma 1.64 to be valid until time $t_k - t_{k-1}$. Next one needs to choose γ_{k-1} sufficiently small such that one can run the Evolution Equation again until time $t_{k-1} - t_{k-2}$ started from *any* function $\gamma_{k-1} u \in \mathcal{C}_{\mathfrak{d}}^{\alpha+2+\delta/8}$ such that $\|u\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2+\delta/8}} \leq \|\phi_{k-1}\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2+\delta/8}} + 1$. Now,

$$\left\| \phi_{k-1} + \frac{1}{\gamma_{k-1}} U_{t_k - t_{k-1}}(\gamma_k \phi_k) \right\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2+\delta/8}} \leq \|\phi_{k-1}\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2+\delta/8}} + \frac{1}{\gamma_{k-1}} \|U_{t_k - t_{k-1}}(\gamma_k \phi_k)\|_{\mathcal{C}_{\mathfrak{d}}^{\alpha+2+\delta/8}}.$$

Using (13), we can now refine our choice for γ_k such that the second term is bounded by 1. In a similar fashion, we can carry on the iteration by subsequently refining already established coefficients. Using the injectivity of the Laplace transform, [Kal83, 15.5.1], we get that the distributions of $(\langle \phi_1, X_{t_1}^m \rangle, \ldots, \langle \phi_k, X_{t_k}^m \rangle)$ are uniquely determined. Using that $M = \{\langle \phi, \cdot \rangle \,|\, \phi \in C_c^\infty((0,L)^2), \phi \geq 0\}$ is a separating set of functions, we get that the distribution of $(X_{t_1}^m, \ldots, X_{t_k}^m)$ is unique after an application of [EK86, Proposition 3.4.6].
Let D be the set containing the Borel measurable subsets A of $D([0,\infty), M((0,L)^2))$ on which $\mathrm{Law}(X^m)(A)$ is uniquely determined. It follows by the continuity from below of $\mathrm{Law}(X^m)$, that D is a Dynkin class.
Every finite dimensional cylinder set can be written as the intersection of sets $\{X_t^m \in A\}$, where $A \subset M((0,L)^2)$ is a Borel set and $t \geq 0$. By [EK86, Proposition 3.7.1] it follows that each such set is a Borel subset of $D([0,\infty), M((0,L)^2))$, hence each finite dimensional cylinder set is a Borel set as well.
On the other hand, the set of finite-dimensional cylinders C is intersection stable and we have shown that $C \subset D$. Consequently by the Dynkin Class theorem, [EK86, Appendix, Theorem 4.2], $\sigma(X_t^m, 0 \leq t < \infty) = \sigma(C) \subset D$.

□

2.4 Continuity of the Killed Mollified Super Brownian Motion

In this section we prove the continuity of the killed mollified SBM. For this we need another result:

Theorem 2.42 [EK86, Theorem 3.10.2]
Let (E, r) be a metric space. For $x \in D([0,\infty), E)$, we define

$$J(x) = \int_0^\infty \exp(-u)(J(x,u) \wedge 1)du,$$

where $J(x,u) := \sup_{0 \leq t \leq u} r(x(t), x(t-))$. Let $(Z_n)_{n \in \mathbb{N}}$ and Z be processes in $D([0,\infty), E)$ such that $Z_n \to Z$ weakly. Then Z is a.s. continuous if and only if $J(Z_n) \to 0$ weakly as $n \to \infty$.

We can now apply Theorem 2.42:

Lemma 2.43 Cf. [PR19, Lemma 4.4]
The killed mollified Super Brownian Motion X^m started in $\mu_m \in M_F((0,L)^2)$ is almost surely continuous.

Proof
Let $(f_k)_{k\in\mathbb{N}}$ be as in Theorem 2.25. Since for any $x \geq 0$, $1-x \leq \exp(-x) \leq 1$, we get that almost surely,

$$\begin{aligned}
J(X^{n,m}) &= \int_0^\infty \exp(-u)\left(\sup_{0\leq t\leq u}\sum_{k=1}^\infty 2^{-k}(1-\exp(-|\langle f_k, X_t^{n,m}\rangle - \langle f_k, X_{t-}^{n,m}\rangle|)) \wedge 1\right) du \\
&\leq \int_0^\infty \exp(-u)\left(\sum_{k=1}^\infty \sup_{0\leq t\leq u} 2^{-k}|\langle f_k, X_t^{n,m}\rangle - \langle f_k, X_{t-}^{n,m}\rangle| \wedge 1\right) du \\
&\leq \int_0^\infty \exp(-u)\left(\sum_{k=1}^\infty 2^{-k}\frac{1}{n}\|f_k\|_{L^\infty} \wedge 1\right) du \\
&\leq \frac{1}{n}\int_0^\infty \exp(-u) du.
\end{aligned}$$

Therefore $J(X^{n,m}) \to 0$ almost surely as $n \to \infty$, which yields the claim.

□

3 Construction of the Killed Rough Super Brownian Motion

In this section we construct the killed rough Super Brownian Motion as the limit of the killed mollified Super Brownian Motion as $m \to \infty$. This tells us that the kmSBM is close to the krSBM for large m, a fact which we will again encounter in Section 4. By doing so, we also establish that the kmSBM falls in the universality class of the krSBM, establishing it as the first non-discrete process to do so.

3.1 The White Noise Limit: Killed Rough Super Brownian Motion

Before we proceed, we need to fix a suitable domain for our limiting martingale problem. The fact that we work with paracontrolled distributions now forces us to be careful, since pointwise arguments rarely carry over.

Definition 3.1 [PR19, p. 12]
Let $D_{\mathscr{H}_\xi}$ be as in Definition 1.59. We define the martingale problem for $(L, D(L))$ by $D(L) := \{\exp(-\langle\phi,\cdot\rangle) | \phi \in D_{\mathscr{H}_\xi}\}$, where for $\phi \in D_{\mathscr{H}_\xi}$ the generator is given by

$$L\exp(-\langle\phi,\cdot\rangle)(\mu) = \left\langle -\mathscr{H}_\xi\phi + \frac{1}{2}\phi^2, \mu\right\rangle \exp(-\langle\phi,\mu\rangle).$$

We call any solution to the this martingale problem a killed rough Super Brownian Motion.

Remark 3.2
As established in Section 1.7, the crucial observation is that for $u_0 \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$, $u_0^m \in I_{\mathrm{PAM}}^{\mathrm{m}}$,

$\mathscr{H}_{\xi} A_t u_0 = T_t u_0 - u_0 \in \mathcal{C}_{\mathfrak{d}}^{\alpha+2}$ and $\mathscr{H}_{\xi_m} A_t^m u_0^m = T_t^m u_0^m - u_0^m \in \mathcal{C}_{\mathfrak{d}}^{\alpha+2}$. Therefore, the convergence of $\mathscr{H}_{\xi_m} A_t^m u_0^m \to \mathscr{H}_{\xi} A_t u_0$ can be achieved not just in L^2, cf. Section 1.4, but also in $\mathcal{C}_{\mathfrak{d}}^{\alpha+2}$, if $u_0^m \to u_0$ in $\mathcal{C}_{\mathfrak{d}}^{\alpha+2}$. By Lemma 1.56, it holds that $T_t u_0 \geq 0$ if $u_0 \geq 0$, $u_0 \neq 0$, and consequently $A_t u_0 \geq 0$. What is more for any $u_0 \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$, $u_0 \geq 0$, $u_0 \neq 0$, $1/t A_t u_0 \to u_0$ as $t \downarrow 0$ in $\mathcal{C}_{\mathfrak{d}}^{\alpha+2}$ by Lemma 1.60.

We will once more apply Theorem 2.22. To do so, we need to establish first some uniform bounds. This can be done by applying our preliminary bounds and Lemma 2.39 to achieve explicit moment expressions.

Lemma 3.3
Let $m \in \mathbb{N}$ and $\phi \in C^1([0,\infty), C_0((0,L)^2))$, $\phi(t) \in Dom(\mathscr{H}_{\xi_m})$ for any $t \geq 0$ and $\phi \geq 0$. Assume that $\mathscr{H}_{\xi_m}\phi \in C([0,\infty), C_0((0,L)^2))$. Then it holds that

$$\tilde{L}_m^{\phi}(t) := \langle \phi(t), X_t^m \rangle - \langle \phi(0), X_0^m \rangle - \int_0^t \langle \mathscr{H}_{\xi_m}\phi(s) + \partial_s\phi(s), X_s^m \rangle \, ds$$

is a continuous, square-integrable martingale with quadratic variation

$$\langle \tilde{L}_m^{\phi} \rangle_t = \int_0^t \langle \phi^2(s), X_s^m \rangle ds.$$

What is more, we have the martingales

$$\begin{aligned} \tilde{M}_m^{1,\phi}(t) &:= \langle \phi(t), X_t^m \rangle^2 - \langle \phi(0), X_0^m \rangle^2 \\ &- \int_0^t \langle \phi^2(s), X_s^m \rangle + 2 \langle \mathscr{H}_{\xi_m}\phi(s) + \partial_s\phi(s), X_s^m \rangle \langle \phi(s), X_s^m \rangle \, ds, \\ \tilde{M}_m^{2,\phi}(t) &:= \left(\langle \phi(t), X_t^m \rangle - \int_0^t \langle \mathscr{H}_{\xi_m}\phi(s) + \partial_s\phi(s), X_s^m \rangle \, ds \right) \langle \phi(0), X_0^m \rangle, \end{aligned}$$

and the local martingale

$$\begin{aligned} &\tilde{M}_m^{3,\phi}(t) := 2 \int_0^t \langle \mathscr{H}_{\xi_m}\phi(s) + \partial_s\phi(s), X_s^m \rangle \langle \phi(s), X_s^m \rangle \, ds \\ &+ \left(\int_0^t \langle \mathscr{H}_{\xi_m}\phi(s) + \partial_s\phi(s), X_s^m \rangle \, ds \right)^2 - 2 \langle \phi(t), X_t^m \rangle \int_0^t \langle \mathscr{H}_{\xi_m}\phi(s) + \partial_s\phi(s), X_s^m \rangle \, ds. \end{aligned}$$

Proof
The claim follows as in Lemma 2.21 using Lemma 2.39 and Lemma 2.38, the computations are however simpler, due to the absence of non-trivial fractions in the generator.
□

We can argue as before to remove the Anderson Hamiltonian:

Lemma 3.4
It holds that for $s \leq t$ and $\phi \in Dom(\mathscr{H}_{\boldsymbol{\xi}_m})$, $\phi \geq 0$,

$$L_m^{\phi}(s,t) := \langle T_{t-s}^m \phi, X_s^m \rangle - \langle T_t^m \phi, X_0^m \rangle$$

is a continuous, square-integrable martingale with quadratic variation given by

$$\langle L^{\phi}(\cdot,t)\rangle_s = \int_0^s \langle (T_{t-r}^m \phi)^2, X_r^m \rangle dr.$$

What is more,

$$M_m^{1,\phi}(s,t) := \langle T_{t-s}^m \phi, X_s^m \rangle^2 - \langle T_t^m \phi, X_0^m \rangle^2 - \int_0^s \langle (T_{t-r}^m \phi)^2, X_r^m \rangle dr$$

is a martingale.

Proof
The proof is a direct consequence of Lemma 3.3 and Theorem 1.41 using the martingales $\tilde{L}_m^{\phi}(t)$ and $\tilde{M}_m^{1,\phi}(t)$.

□

One way of establishing moment bounds that are uniform in m is to use preliminary bounds to show explicit moment expressions. Here, the evolution equation for the killed mollified SBM and the Wild sum representation of its solution comes into play.

Lemma 3.5
Assume that $\sup_{m\in\mathbb{N}} \langle \mathbb{1}_{(0,L)^2}, \mu_m \rangle < \infty$*. Then it holds that for any $T > 0$,*

$$\sup_{0\leq t\leq T} \sup_{m\in\mathbb{N}} \mathbb{E}\left(\left\langle \mathbb{1}_{(0,L)^2}, X_t^m \right\rangle^4 \right) < \infty.$$

Proof
Let $\phi \in \mathrm{Dom}(\mathscr{H}_{\boldsymbol{\xi}_m}) \cap \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$, $\phi \geq 0$, $\phi \neq 0$ and $\gamma > 0$ be sufficiently small. We have by Lemma 2.40 and the results of Section 1.9 and 1.10 the representation

$$\mathbb{E}\left(\exp\left(-\gamma \langle \phi, X_t^m \rangle\right)\right) = \exp\left(- \left\langle \sum_{n=1}^{\infty} a_n^m(t,\cdot)\gamma^n, \mu_m \right\rangle \right) =: f(\gamma),$$

for certain functions a_n^m which also depend on ϕ. By Lemma 2.38, we are allowed to exchange differentiation and integration and subsequently let $\gamma \downarrow 0$ in the above. We can now compute:

$$\begin{aligned}
\frac{d}{d\gamma} f(\gamma) &= -\left\langle \sum_{n=1}^{\infty} a_n^m(t,\cdot)\gamma^{n-1} n, \mu_m \right\rangle f(\gamma),\\
\frac{d^2}{d\gamma^2} f(\gamma) &= \left\langle \sum_{n=1}^{\infty} a_n^m(t,\cdot)\gamma^{n-1} n, \mu_m \right\rangle^2 f(\gamma) - \left\langle \sum_{n=2}^{\infty} a_n^m(t,\cdot)\gamma^{n-2} n(n-1), \mu_m \right\rangle f(\gamma),\\
\frac{d^3}{d\gamma^3} f(\gamma) &= 3\left\langle \sum_{n=1}^{\infty} a_n^m(t,\cdot)\gamma^{n-1} n, \mu_m \right\rangle \left\langle \sum_{n=2}^{\infty} a_n^m(t,\cdot)\gamma^{n-2} n(n-1), \mu_m \right\rangle f(\gamma)\\
&- \left\langle \sum_{n=1}^{\infty} a_n^m(t,\cdot)\gamma^{n-1} n, \mu_m \right\rangle^3 f(\gamma) - \left\langle \sum_{n=3}^{\infty} a_n^m(t,\cdot)\gamma^{n-3} n(n-1)(n-2), \mu_m \right\rangle f(\gamma),
\end{aligned}$$

and

$$\begin{aligned}
\frac{d^4}{d\gamma^4} f(\gamma) &= 3\left\langle \sum_{n=2}^{\infty} a_n^m(t,\cdot)\gamma^{n-2} n(n-1), \mu_m \right\rangle^2 f(\gamma)\\
&+ 4\left\langle \sum_{n=1}^{\infty} a_n^m(t,\cdot)\gamma^{n-1} n, \mu_m \right\rangle \left\langle \sum_{n=3}^{\infty} a_n^m(t,\cdot)\gamma^{n-3} n(n-1)(n-2), \mu_m \right\rangle f(\gamma)\\
&- 6\left\langle \sum_{n=1}^{\infty} a_n^m(t,\cdot)\gamma^{n-1} n, \mu_m \right\rangle^2 \left\langle \sum_{n=2}^{\infty} a_n^m(t,\cdot)\gamma^{n-2} n(n-1), \mu_m \right\rangle f(\gamma)\\
&+ \left\langle \sum_{n=1}^{\infty} a_n^m(t,\cdot)\gamma^{n-1} n, \mu_m \right\rangle^4 f(\gamma) - \left\langle \sum_{n=4}^{\infty} a_n^m(t,\cdot)\gamma^{n-4} n(n-1)(n-2)(n-3), \mu_m \right\rangle f(\gamma).
\end{aligned}$$

Letting $\gamma = 0$ yields

$$\begin{aligned}
\frac{d^4}{d\gamma^4} f(0) &= 12\left\langle a_2^m(t,\cdot), \mu_m\right\rangle^2 + 24\left\langle a_1^m(t,\cdot), \mu_m\right\rangle \left\langle a_3^m(t,\cdot), \mu_m\right\rangle\\
&\quad - 12\left\langle a_1^m(t,\cdot), \mu_m\right\rangle^2 \left\langle a_2^m(t,\cdot), \mu_m\right\rangle + \left\langle a_1^m(t,\cdot), \mu_m\right\rangle^4 - 24\left\langle a_4^m(t,\cdot), \mu_m\right\rangle .
\end{aligned}$$

Since $\mathbb{E}(\langle \phi, X_t^m\rangle^4) = \partial_\gamma^4 f(0)$, we only need to identify the functions $a_1^m, \dots, a_4^m$ and show that they are bounded uniformly in m. It turns out that

$$\begin{aligned}
a_1^m(t,x) &= \bullet = T_t^m \phi(x), \quad a_2^m(t,x) = \vee = -\frac{1}{2}\int_0^t T_s^m (T_{t-s}^m \phi)^2(x) ds,\\
a_3^m(t,x) &= 2 \times \tau_3 = -\int_0^t T_s^m \left(\left(-\frac{1}{2}\int_0^{t-s} T_r^m (T_{t-s-r}^m \phi)^2 dr \right) T_{t-s}^m \phi \right)(x) ds,\\
a_4^m(t,x) &= \tau_4 + 4 \times \tau_4' = -\frac{1}{2}\int_0^t T_s^m \left(-\frac{1}{2}\int_0^{t-s} T_r^m (T_{t-s-r}^m \phi)^2 dr \right)^2 (x) ds\\
&- 2\int_0^t T_s^m \left(-\frac{1}{2}\int_0^{t-s} T_r^m \left(\left(-\frac{1}{2}\int_0^{t-s-r} T_u^m (T_{t-s-r-u}^m \phi)^2 du \right) T_{t-s-r}^m \phi \right) dr T_{t-s}^m \phi \right)(x) ds.
\end{aligned}$$

We get by the maximum principle $T_s^m (T_{t-s}^m \phi)^2 \le \|T_{t-s}^m \phi\|_{L^\infty} T_t^m \phi \le \|T_{t-s}^m \phi\|_{L^\infty} \|T_t^m \phi\|_{L^\infty}$ and similar bounds for the other terms. It follows by Lemma 1.53 that the RHSs are

bounded uniformly in m and $t \leq T$, which yields the first claim.
For the second claim, let as in Lemma 2.38, $X^m_{(0,L)^2} \leq X^m_{(-L-1,L+1)^2}$ be coupled killed mollified SBMs on $(0,L)^2$ and $(-L-1,L+1)^2$ respectively. Let $\phi \in C^2_c((-L-1,L+1)^2)$, $\phi \geq 0$, be such that $\mathbb{1}_{(0,L)^2} \leq \phi$. Then, $\langle \mathbb{1}_{(0,L)^2}, X^m_{(0,L)^2}(t)\rangle \leq \langle \phi, X^m_{(-L-1,L+1)^2}(t)\rangle$ and the fourth moment of the RHS is bounded uniformly in m and $t \leq T$ by the above. This yields the claim.

□

We now show the tightness of the killed mollified SBMs $(X^m)_{m\in\mathbb{N}}$.

Lemma 3.6
Let for $m \in \mathbb{N}$, X^m be a solution to the martingale problem for $(L_m, D(L_m))$ started in μ_m such that $\sup_{m\in\mathbb{N}}\langle \mathbb{1}_{(0,L)^2}, \mu_m\rangle < \infty$. Then $(X^m)_{m\in\mathbb{N}}$ is tight.

Proof
We first establish tightness of the evaluated processes by an application of Theorem 2.31 and then deduce the compact containment condition. Tightness of the measure-valued processes then follows by Jakubowski's criterion, Theorem 2.29.
Let $\phi \in C^2_c((0,L)^2)$, $\phi \geq 0$. For any $t > 0$ and $K > 0$ it holds that

$$\mathbb{P}(\langle \phi, X^m_t\rangle > K) \leq \frac{1}{K}\mathbb{E}(\langle \phi, X^m_t\rangle) \leq \frac{1}{K}\sup_{m\in\mathbb{N}}\mathbb{E}(\langle \phi, X^m_t\rangle).$$

Condition (20) now follows by Lemma 3.5. We compute for $\delta > 0$,

$$\begin{aligned}\mathbb{E}\left(\langle \phi, X^m_\delta - X^m_0\rangle^2\right) &= \mathbb{E}\left(\left|L^\phi_m(\delta,\delta) - L^\phi_m(0,\delta) + \langle T^m_\delta\phi - \phi, \mu_m\rangle\right|^2\right)\\ &\lesssim \int_0^\delta \mathbb{E}\left(\left\langle (T^m_{\delta-r}\phi)^2, X^m_r\right\rangle\right) dr + \langle T^m_\delta \phi - \phi, \mu_m\rangle^2 .\end{aligned}$$

We have for $\delta < 1$,

$$\sup_{m\in\mathbb{N}}\int_0^\delta \mathbb{E}\left(\left\langle (T^m_{\delta-r}\phi)^2, X^m_r\right\rangle\right) dr \leq \delta \sup_{m\in\mathbb{N}}\sup_{r\leq 1}\|T^m_r\phi\|^2_{L^\infty}\sup_{m\in\mathbb{N}}\sup_{0\leq r\leq 1}\mathbb{E}(\langle \mathbb{1}_{(0,L)^2}, X^m_r\rangle)$$

and by Lemma 1.53,

$$\langle T^m_\delta\phi - \phi, \mu_m\rangle^2 \leq \delta^{\alpha+2}(C^m_1\exp(C^m_2)\|\phi\|_{\mathcal{C}^{\alpha+2}_\mathfrak{d}})^2\langle \mathbb{1}_{(0,L)^2}, \mu_m\rangle^2.$$

It follows that

$$\lim_{\delta\to 0}\sup_{m\in\mathbb{N}}\mathbb{E}\left(\langle \phi, X^m_\delta - X^m_0\rangle^2\right) = 0. \tag{26}$$

Next we get

$$\begin{aligned}&\mathbb{E}(\langle \phi, X^m_{t+u} - X^m_t\rangle^2\,|\mathcal{F}^m_t)\\ &= \mathbb{E}\left(\left|L^\phi_m(t+u,t+u) - L^\phi_m(t,t+u) + \langle T^m_u\phi - \phi, X^m_t\rangle\right|^2\Big|\mathcal{F}^m_t\right)\\ &\lesssim \int_t^{t+u}\mathbb{E}(\langle (T^m_{t+u-r}\phi)^2, X^m_r\rangle|\mathcal{F}^m_t)dr + \langle T^m_u\phi - \phi, X^m_t\rangle^2 .\end{aligned} \tag{27}$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}\left((\langle \phi, X^m_{t+u} - X^m_t \rangle \wedge 1)^2 \, (\langle \phi, X^m_t - X^m_{t-u} \rangle \wedge 1)^2 \right) \\
&\lesssim \mathbb{E}\left(\left(\int_t^{t+u} \mathbb{E}(\langle (T^m_{t+u-r}\phi)^2, X^m_r \rangle | \mathcal{F}^m_t) dr \right)^2 \right)^{1/2} \mathbb{E}\left(\langle \phi, X^m_t - X^m_{t-u} \rangle^2 \right)^{1/2} \\
&+ \mathbb{E}\left(\langle T^m_u \phi - \phi, X^m_t \rangle^4 \right)^{1/2} \mathbb{E}\left(\langle \phi, X^m_t - X^m_{t-u} \rangle^2 \right)^{1/2}.
\end{aligned}
$$

By taking the expectation of (27), uniformly in m,

$$
\mathbb{E}\left(\langle \phi, X^m_t - X^m_{t-u} \rangle^2 \right)^{1/2} \lesssim \left(u + u^{\alpha+2} \right)^{1/2}.
$$

As above this yields that uniformly in m,

$$
\begin{aligned}
&\mathbb{E}\left((\langle \phi, X^m_{t+u} - X^m_t \rangle \wedge 1)^2 \, (\langle \phi, X^m_t - X^m_{t-u} \rangle \wedge 1)^2 \right) \\
&\lesssim u \left(u + u^{\alpha+2} \right)^{1/2} + u^{\alpha+2} \left(u + u^{\alpha+2} \right)^{1/2}.
\end{aligned}
$$

The decay in u is of order greater than 1, which yields by Theorem 2.31 the tightness of $\langle \phi, X^m \rangle$, $\phi \in C^2_c((0,L)^2)$, $\phi \geq 0$.
In order to establish the compact containment condition, we argue as follows. By the tightness for $\langle \phi, X^m \rangle$ established above, there exists for any $\varepsilon > 0$ some $C(\phi, \varepsilon) \subset D([0,\infty), \mathbb{R})$ compact such that for any $m \in \mathbb{N}$, $\mathbb{P}(\langle \phi, X^m \rangle \in C(\phi, \varepsilon)) \geq 1 - \varepsilon$. By [EK86, p. 152, Problem 16], there exists for any $T > 0$ some $K(T, \phi, \varepsilon) > 0$ such that for any $m \in \mathbb{N}$,

$$
\mathbb{P}\left(\sup_{0 \leq t \leq T} \langle \phi, X^m_t \rangle < K(T, \phi, \varepsilon) \right) \geq \mathbb{P}(\langle \phi, X^m \rangle \in C(\phi, \varepsilon)) \geq 1 - \varepsilon.
$$

Let $X^m_{(0,L)^2} \leq X^m_{(-L-1,L+1)^2}$ be coupled, killed mollified SBMs on $(0,L)^2$, $(-L-1, L+1)^2$ respectively. Let $\phi \in C^2_c((-L-1, L+1)^2)$, $\phi \geq 0$, such that $\mathbb{1}_{(0,L)^2} \leq \phi$. Then for any $m \in \mathbb{N}$,

$$
\begin{aligned}
&\mathbb{P}\left(\sup_{0 \leq t \leq T} \langle \mathbb{1}_{(0,L)^2}, X^m_{(0,L)^2}(t) \rangle < K(T, \phi, \varepsilon) \right) \\
&\geq \mathbb{P}\left(\sup_{0 \leq t \leq T} \langle \phi, X^m_{(-L-1,L+1)^2}(t) \rangle < K(T, \phi, \varepsilon) \right) \geq 1 - \varepsilon.
\end{aligned}
$$

Therefore we can choose for any $T, \varepsilon > 0$, $K(T, \varepsilon) := K(T, \phi, \varepsilon) > 0$ such that the compact containment condition holds, using the C_c-vaguely relatively compact set of Lemma 2.30 given by $B = \{\mu \in M((0,L)^2) | \langle \mathbb{1}_{(0,L)^2}, \mu \rangle < K(T, \varepsilon)\}$. Tightness of $(X^m)_{m \in \mathbb{N}}$ now follows by Jakubowski's criterion, Theorem 2.29, using that the functions $\mu \mapsto \langle \phi, \mu \rangle$ with $\phi \in C^2_c((0,L)^2)$, $\phi \geq 0$, are continuous by the definition of the C_c-vague topology, closed under addition, and also separating points.

□

We can now show the convergence of the generators on subsubsequences.

Theorem 3.7
Conditions (16)*,* (17) *and* (18) *of Theorem 2.22 hold for the martingale problems for* $((L_m, D(L_m)))_{m\in\mathbb{N}}$ *and* $(L, D(L))$ *started in* $\mu_m, \mu \in M_F((0,L)^2)$ *respectively such that* $\sup_{m\in\mathbb{N}}\langle \mathbb{1}_{(0,L)^2}, \mu_m\rangle < \infty$ *and* $\mu_m \to \mu$ *in* $M((0,L)^2)$*.*

Proof
The proof is similar to, but easier than, the proof of Theorem 2.37, now using Lemma 3.5, Lemma 3.6 and (11).

□

To complete the construction, we need to show uniqueness of the limiting martingale problem. Before we do so, we again need to establish some moment bounds in order to find more martingales. Also, we impose the same assumption as before:

Assumption: Whenever we write $\mu \in M_F((0,L)^2)$ as the starting point of a killed rough SBM, we implicitly assume that there exist some $\mu_m \in M_F((0,L)^2)$, $m \in \mathbb{N}$, such that it holds that $\sup_{m\in\mathbb{N}}\langle \mathbb{1}_{(0,L)^2}, \mu_m\rangle < \infty$ and $\mu_m \to \mu$, C_c-vaguely as $m \to \infty$.

3.2 Moment Bounds for the Killed Rough Super Brownian Motion

We recall that convergence of the generators and relative compactness is already enough to assert that for some subsequence $X^m \to X$ as $m \to \infty$, where X is a solution to the martingale problem for $(L, D(L))$. A moment bound for X is given by the following:

Lemma 3.8
Let X *be a solution to the martingale problem for* $(L, D(L))$ *started in* $\mu \in M_F((0,L)^2)$*. Then it holds that for* $T > 0$*,*

$$\sup_{0\le t\le T} \mathbb{E}\left(\left\langle \mathbb{1}_{(0,L)^2}, X_t\right\rangle^4\right) < \infty.$$

Also, X *is actually a finite measure a.s..*

Proof
The proof follows exactly as in Lemma 2.38.

□

3.3 Continuity and Uniqueness of the Killed Rough Super Brownian Motion

In this section we show the continuity and uniqueness of solutions to the martingale problem for the killed rough Super Brownian Motion.
Continuity follows immediately from the continuity of the killed mollified SBM.

Lemma 3.9
Let X *be a killed rough Super Brownian Motion. Then* X *is almost surely continuous.*

Proof
Let $(X^m)_{m\in\mathbb{N}}$ be a sequence of killed mollified Super Brownian Motions such that $X^m \to X$ weakly. By Theorem 2.42 it suffices to show that $J(X^m) \to 0$ weakly as $m \to \infty$. However, X^m is almost surely continuous, therefore the LHS is 0 and the claim follows. □

Next we show the uniqueness of the killed rough SBM by extending the methods presented in [PR19] to our setting. Due to the singularity of the white noise and the Dirichlet boundary condition, the argument is more involved than the one for the killed mollified SBM.

Lemma 3.10
Our killed rough Super Brownian Motion coincides with the one from [Ros19, Theorem 3.8]. In particular for any $\phi \in C^1([0,\infty), D_{\mathscr{H}_\xi})$ such that $\mathscr{H}_\xi\phi \in C([0,\infty), \mathcal{C}_{\mathfrak{d}}^{\alpha+2})$,

$$\tilde{L}^\phi(t) := \langle \phi(t), X_t\rangle - \langle \phi(0), X_0\rangle - \int_0^t \langle \mathscr{H}_\xi\phi(s) + \partial_s\phi(s), X_s\rangle\, ds$$

is a continuous, square-integrable martingale with quadratic variation

$$\langle \tilde{L}^\phi\rangle_t = \int_0^t \langle \phi(s)^2, X_s\rangle ds.$$

Proof
Using the bounds established in Lemma 3.8, we arrive at the claim exactly as in Lemma 3.3. □

Next we extend this martingale problem to solutions to the backwards PAM with forcing:

Lemma 3.11
Let $t > 0$, $u_0 \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$, $u_0 \geq 0$, $u_0 \neq 0$, and $f \in C([0,t], \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+})$, $f \geq 0$, $f \neq 0$. Let u^t be a mild solution to

$$\begin{cases} (\partial_s + \mathscr{H}_\xi)u^t = f \text{ in } (0,t)\times(0,L)^2, \\ u^t(t) = u_0 \text{ in } [0,L]^2,\ u^t = 0 \text{ on } [0,t]\times\partial[0,L]^2, \end{cases}$$

as constructed in Section 1.8. Then it holds that for $s \leq t$,

$$L^{u_0,f}(s,t) := \langle u^t(s), X_s\rangle - \langle u^t(0), X_0\rangle - \int_0^s \langle f(r), X_r\rangle dr,$$

is a continuous, square-integrable martingale with quadratic variation

$$\langle L^{u_0,f}(\cdot,t)\rangle_s = \int_0^s \langle (u^t(r))^2, X_r\rangle dr.$$

Proof
First let u_0^k, f^k and u_k^t be as in Section 1.8. Then u_k^t is a pointwise solution to the backwards PAM with forcing, hence by Lemma 3.10,

$$L^{u_0^k,f^k}(s,t) := \langle u_k^t(s), X_s\rangle - \langle u_k^t(0), X_0\rangle - \int_0^s \langle f^k(r), X_r\rangle dr,$$

is a continuous, square-integrable martingale with quadratic variation

$$\langle L^{u_0^k,f^k}(\cdot,t)\rangle_s = \int_0^s \langle (u_k^t(r))^2, X_r\rangle dr.$$

We have shown that $u_k^t \to u^t$ in $C([0,t], \mathcal{C}_{\mathfrak{d}}^{\alpha+2})$. Therefore the claim follows by [EK86, p. 174, Equation (3.4)], the dominated convergence theorem and the bounds of Lemma 3.8.

□

With the martingale problem above we can find exponential martingales by an application of Itô's formula:

Lemma 3.12
Let $\psi \in \mathcal{C}_{\mathfrak{d}}^{(\alpha+2)+}$, $\psi \geq 0$, $\psi \neq 0$, $t > 0$, and $\gamma > 0$ be sufficiently small. Then it holds that the process for $s \leq t$ given by

$$E^{\gamma\psi}(s,t) := \exp(-\langle U_{t-s}(\gamma\psi), X_s\rangle) - \exp(-\langle U_t(\gamma\psi), X_0\rangle)$$

is a continuous, bounded martingale.

Proof
Let u^t, u_0 and f be as in Lemma 3.11. We apply Itô's formula to $\exp(-x)$ to get

$$\begin{aligned}
&\exp(-\langle u^t(s), X_s\rangle) - \exp(-\langle u^t(0), X_0\rangle)\\
&= -\int_0^s \exp(-\langle u^t(r), X_r\rangle) dL^{u_0,f}(r,t) - \int_0^s \exp(-\langle u^t(r), X_r\rangle)\langle f(r), X_r\rangle dr\\
&+ \frac{1}{2}\int_0^s \exp(-\langle u^t(r), X_r\rangle)\langle (u^t(r))^2, X_r\rangle dr.
\end{aligned}$$

By plugging in $f(r) = 1/2(U_{t-r}(\gamma\psi))^2$, $u^t(r) = U_{t-r}(\gamma\psi)$, we get that

$$\begin{aligned}
E^{\gamma\psi}(s,t) &= \exp(-\langle U_{t-s}(\gamma\psi), X_s\rangle) - \exp(-\langle U_t(\gamma\psi), X_0\rangle)\\
&= -\int_0^s \exp(-\langle U_{t-r}(\gamma\psi), X_r\rangle) dL^{u_0,f}(r,t)
\end{aligned}$$

is a local martingale and a proper martingale by the conjectured non-negativity of $U(\gamma\psi)$.
□

The uniqueness of the killed rough Super Brownian Motion now follows as before:

Lemma 3.13
Any solution to the martingale problem for $(L, D(L))$ started in $\mu \in M_F((0,L)^2)$ is unique.

Proof
The proof follows as in Lemma 2.41 using Lemma 3.12.

□

4 Persistence of the Killed Mollified Super Brownian Motion

In this section we prove the persistence of the killed mollified Super Brownian Motion for sufficiently large m and L. This is based on a spectral decomposition using the Eigenfunctions of $\mathscr{H}_{\boldsymbol{\xi}_m}$, the fact that the principal Eigenvalue is positive for m and L large enough and the Krein-Rutman theorem. Those ideas were inspired by [PR19, Corollary 5.6].

Lemma 4.1
The operator $\mathscr{H}_{\boldsymbol{\xi}_m} : H_0^2 \to L^2$ admits the Eigenfunctions $(u_k(\boldsymbol{\xi}_m))_{k\in\mathbb{N}}$ and Eigenvalues $(\lambda_k(\boldsymbol{\xi}_m))_{k\in\mathbb{N}}$. The Eigenfunctions furthermore lie in Dom$(\mathscr{H}_{\boldsymbol{\xi}_m})$.

Proof
The first claim follows from Theorem 1.29 using that $\mathfrak{D}^{\mathfrak{d},\gamma}_{\boldsymbol{\xi}_m} = H_0^2$. Assume $u \in H_0^2$, then $\tilde{u} \in B^2_{2,2}$ and it follows from the Besov embedding theorem, [GIP15, Lemma A.2], that $\tilde{u} \in \mathcal{C}^1$. Consequently, $u \in C_0((0,L)^2)$. Next, we have for any $k \in \mathbb{N}$, at least a.e.,

$$\begin{cases} \mathscr{H}_{\boldsymbol{\xi}_m} u_k(\boldsymbol{\xi}_m) = \lambda_k(\boldsymbol{\xi}_m) u_k(\boldsymbol{\xi}_m) & \text{in } (0,L)^2, \\ u_k(\boldsymbol{\xi}_m) = 0 \quad \text{on } \partial[0,L]^2. \end{cases}$$

It follows by [Eva10, Theorem 6.3.1.3], that $u_k(\boldsymbol{\xi}_m) \in C^2((0,L)^2)$. One may characterise

$$C_0((0,L)^2) = \{f \in C((0,L)^2) | \forall \varepsilon > 0 \ \exists K \subset (0,L)^2 \text{ compact } f \restriction_{(0,L)^2 \setminus K} < \varepsilon\}.$$

This yields that $\mathscr{H}_{\boldsymbol{\xi}_m} u_k(\boldsymbol{\xi}_m) \in C_0((0,L)^2)$ as well.

□

Next we show that the Eigenfunction $u_1(\boldsymbol{\xi}_m)$ to the principal Eigenvalue $\lambda_1(\boldsymbol{\xi}_m)$ is positive in the interior.

Lemma 4.2
It holds that $u_1(\boldsymbol{\xi}_m) > 0$ in $(0,L)^2$.

Proof
We can choose by [CvZ19, Lemma 5.12], $A \geq \max\{\sup_{x\in[0,L]^2} |\xi_m(x) - c_m|, \lambda_1(\boldsymbol{\xi}_m)\}$

sufficiently large such that $(A - \mathscr{H}_{\boldsymbol{\xi}_m})^{-1} : L^2 \to H_0^2$ exists, is self-adjoint and compact. Let $K = \{f \in L^2((0,L)^2) | f \geq 0 \text{ a.e.}\}$. This set is a total cone by [Dei85, Example 19.2]. In order to apply the Krein-Rutman theorem, [Dei85, Theorem 19.2], we first show that $(\mathscr{H}_{\boldsymbol{\xi}_m} - A)^{-1} K \subset K$.
Proof of the claim: Let $f \in K$ and $(A - \mathscr{H}_{\boldsymbol{\xi}_m})^{-1} f = g$. Then $g \in H_0^2$ and $(A - \mathscr{H}_{\boldsymbol{\xi}_m}) g = f$. Let $\phi \in C_c^\infty((0,L)^2)$. By the definition of weak derivatives,

$$\langle \nabla g, \nabla \phi \rangle_{L^2} + \langle (-(\xi_m - c_m) + A) g, \phi \rangle_{L^2} = \langle f, \phi \rangle_{L^2} .$$

By approximation, the identity carries over to $\phi \in H_0^1$. Let $\phi \geq 0$, then

$$\langle \nabla g, \nabla \phi \rangle_{L^2} + \langle (-(\xi_m - c_m) + A) g, \phi \rangle_{L^2} \geq 0.$$

By [GT01, Theorem 8.1], it follows that $g \leq 0$ everywhere. Consequently, $(\mathscr{H}_{\boldsymbol{\xi}_m} - A)^{-1} f = -g \geq 0$, which proves the claim.
The principal Eigenfunction of $(\mathscr{H}_{\boldsymbol{\xi}_m} - A)^{-1}$ is $u_1(\boldsymbol{\xi}_m)$, hence by the Krein-Rutman theorem, $u_1(\boldsymbol{\xi}_m) \geq 0$ a.e. and by regularity, $u_1(\boldsymbol{\xi}_m) \geq 0$ everywhere. We get $(\mathscr{H}_{\boldsymbol{\xi}_m} - A) u_1(\boldsymbol{\xi}_m) = (\lambda_1(\boldsymbol{\xi}_m) - A) u_1(\boldsymbol{\xi}_m) \leq 0$, since $(\lambda_1(\boldsymbol{\xi}_m) - A) \leq 0$. By [GT01, Theorem 9.6], $-u_1(\boldsymbol{\xi}_m)$ cannot achieve a non-negative maximum in $(0,L)^2$. Consequently, $u_1(\boldsymbol{\xi}_m) > 0$ in $(0,L)^2$.

□

We now establish that the principal Eigenvalue $\lambda_1(\boldsymbol{\xi}_m)$ is positive for m and L sufficiently large.

Lemma 4.3
Let $\lambda > 0$. Then there exist $m, L \in \mathbb{N}$ sufficiently large, $L = 2^n$ for some $n \in \mathbb{N}$, such that $\lambda_1(\boldsymbol{\xi}_m) > \lambda$.

Proof
Let $\lambda_1(\boldsymbol{\xi}, L)$ be the principal Eigenvalue of the Continuous Anderson Hamiltonian $\mathscr{H}_{\boldsymbol{\xi}}$ on $[0,L]^2$. By Theorem 1.30 it holds that a.s. $\lim_{L \to \infty, L = 2^n, n \in \mathbb{N}} \lambda_1(\boldsymbol{\xi}, L)/\log(L) = 2/\rho_1$, with some $\rho_1 > 0$. We assume w.l.o.g. that we consider a realisation $\boldsymbol{\xi}(\omega)$, where this holds.
Note that by [CvZ19, 5.9], it follows that $\mathscr{H}_{\boldsymbol{\xi}_m} = \Delta + \xi_m - c_m$ can indeed be expressed as in Definition 1.26. By Theorem 1.29, there exists some $N > 0$ such that for $\lambda_1(\boldsymbol{\xi}) := \lambda_1(\boldsymbol{\xi}, L)$,$|\lambda_1(\boldsymbol{\xi}) - \lambda_1(\boldsymbol{\xi}_m)| \lesssim \|\boldsymbol{\xi} - \boldsymbol{\xi}_m\|_{\mathfrak{X}_\mathfrak{n}^\alpha} (1 + \|\boldsymbol{\xi}\|_{\mathfrak{X}_\mathfrak{n}^\alpha} + \|\boldsymbol{\xi}_m\|_{\mathfrak{X}_\mathfrak{n}^\alpha})^N$. As $\boldsymbol{\xi}_m \to \boldsymbol{\xi}$ in $\mathfrak{X}_\mathfrak{n}^\alpha$, $\|\boldsymbol{\xi}_m\|_{\mathfrak{X}_\mathfrak{n}^\alpha} \leq \|\boldsymbol{\xi}\|_{\mathfrak{X}_\mathfrak{n}^\alpha} + 1$ for m sufficiently large. Therefore we can choose $L = 2^n$, $n \in \mathbb{N}$, and $m \in \mathbb{N}$ such that $\lambda_1(\boldsymbol{\xi}_m) > \lambda$.

□

Our main result, which states that killed mollified SBM is persistent if m and L are sufficiently large, is the following:

Theorem 4.4 Cf. [PR19, Corollary 5.6]
Let $\lambda > 0$ and let $L' < L$, m be sufficiently large depending on λ. Assume $0 \neq \mu_m \in$

$M_F((0,L)^2)$. It holds that the killed mollified Super Brownian Motion X^m started in μ_m is persistent: For any $\phi \in C_c^\infty((0,L)^2)$ such that $\phi \geq 0$, $\phi \neq 0$, $a+[0,L']^2 \subset\subset Supp(\phi)$ for some $a \in [0,L]^2$, and $\mu(a+[0,L']^2) > 0$, it holds that $\mathbb{P}(\lim_{t\to\infty} e^{-\lambda t} \langle \phi, X_t^m \rangle = \infty) > 0$.

Proof

We denote $\lambda_1 := \lambda_1(\boldsymbol{\xi}_m)$ and $u_1 := u_1(\boldsymbol{\xi}_m)$. We have for $s < t$ by Lemma 3.4, Lemma 4.1 and Lemma 4.2, $\mathbb{E}(\langle u_1, X_t^m\rangle \,|\mathcal{F}_s^m) = \langle T_{t-s}^m u_1, X_s^m\rangle = \langle e^{(t-s)\lambda_1} u_1, X_s^m\rangle$. Therefore, $E_1(s) := \langle e^{-\lambda_1 s} u_1, X_s^m\rangle$ is a martingale. By considering $\tilde{M}_m^{1,\exp(-\lambda_1 \cdot)u_1}$, we get that

$$\mathbb{E}((E_1(t) - E_1(0))^2) = \int_0^t \mathbb{E}(\langle (e^{-\lambda_1 s} u_1)^2, X_s^m\rangle) ds.$$

By an application of the Stone-Weierstrass theorem for locally compact spaces, there exist some $\psi_k \in C_c^\infty((0,L)^2)$, $k \in \mathbb{N}$, such that $\psi_k \to (e^{-\lambda_1 s} u_1)^2$ in $C_0((0,L)^2)$. Naturally $\psi_k \in \mathrm{Dom}(\mathscr{H}_{\boldsymbol{\xi}_m})$ and by Lemma 3.4,

$$\mathbb{E}(\langle (e^{-\lambda_1 s} u_1)^2, X_s^m\rangle) \leftarrow \mathbb{E}(\langle \psi_k, X_s^m\rangle) = \langle T_s^m \psi_k, X_0^m\rangle \to \langle T_s^m((e^{-\lambda_1 s} u_1)^2), X_0^m\rangle.$$

This yields that

$$\mathbb{E}((E_1(t) - E_1(0))^2) = \int_0^t \langle T_s^m (e^{-\lambda_1 s} u_1)^2, X_0^m\rangle ds \leq \int_0^t \|u_1\|_{L^\infty} e^{-\lambda_1 s} \langle u_1, X_0^m\rangle \, ds \lesssim 1.$$

It follows that $E_1(t)$ is an L^2-bounded martingale and hence converges in $L^2(\mathbb{P})$ and almost surely to some random variable $E_1(\infty)$. Specifically, $E_1(\infty) \geq 0$ a.s. and since $\mathbb{E}(E_1(\infty)) = \mathbb{E}(E_1(0)) = \mathbb{E}(\langle u_1, X_0^m\rangle) > 0$, it follows that $\mathbb{P}(E_1(\infty) > 0) > 0$. Let $\phi \in C_c^\infty((0,L)^2)$ be non-negative and not identically zero. Assume that the support of ϕ is large enough, i.e. the compact embedding $a + [0,L']^2 \subset\subset \mathrm{Supp}(\phi)$ holds for some $a \in [0,L]^2$ and L' large enough such that $\lambda_1(\boldsymbol{\xi}_m, L') > \lambda$, with $\lambda_1(\boldsymbol{\xi}_m, L')$ the principal Eigenvalue of $\mathscr{H}_{\boldsymbol{\xi}_m}$ on $a + [0,L']^2$. Let $u_1(\boldsymbol{\xi}_m, L')$ denote the associated Eigenfunction. The translation is harmless by [CvZ19, Lemma 7.4].
Let $C = \sup_{a+(0,L')^2} u_1(\boldsymbol{\xi}_m, L') / \inf_{a+(0,L')^2} \phi$. Then $\mathbb{1}_{a+(0,L')^2} \leq \phi / \inf_{a+(0,L')^2} \phi$ and $u_1(\boldsymbol{\xi}_m, L')/C \leq \mathbb{1}_{a+(0,L')^2} \inf_{a+(0,L')^2} \phi$. Let $X^m_{a+(0,L')^2}$ and $X^m_{(0,L)^2}$ be two coupled, killed mollified SBMs on $a + (0,L')^2$ and $(0,L)^2$ respectively.
We get $1/C \langle u_1(\boldsymbol{\xi}_m, L'), X^m_{a+(0,L')^2}(t)\rangle \leq \langle \phi, X^m_{(0,L)^2}(t)\rangle$. Therefore,

$$\begin{aligned}\mathbb{P}(\lim_{t\to\infty} e^{-\lambda t} \langle \phi, X^m_{(0,L)^2}(t)\rangle = \infty) &\geq \mathbb{P}(\lim_{t\to\infty} e^{-\lambda t} \langle u_1(\boldsymbol{\xi}_m, L'), X^m_{a+(0,L')^2}(t)\rangle = \infty) \\ &\geq \mathbb{P}(E_{1,L'}(\infty) > 0) > 0,\end{aligned}$$

where $E_{1,L'}$ is defined as the E_1 above, now for the killed mollified SBM $X^m_{a+(0,L')^2}$ on $a + (0,L')^2$. This yields the claim for $X^m_{(0,L)^2}$ on $(0,L)^2$.

□

Remark 4.5

In a more recent version of [PR19], Perkowski and Rosati were able to establish the same claim for the killed rough SBM but without the support assumption on ϕ. The version above was suggested by T. Rosati.

5 Index of Notation and Martingales

5.1 Index of Notation

- $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.
- $\mathbb{T}^d_{2L}$, where $L > 0$, $d \in \mathbb{N}$: The torus $(\mathbb{R}/(2L\mathbb{Z}))^d$.
- $C_T X$, $C^\alpha_T X$, where X is some normed space, $T > 0$, $\alpha \in (0, 1]$: The space of (Hölder-) continuous functions $[0, T] \ni t \mapsto v(t) \in X$, equipped with $\|v\|_{C_T X} = \sup_{t \le T} \|v(t)\|_X$ and $\|v\|_{C^\alpha_T X} = \|v\|_{C_T X} + \sup_{0 \le s < t \le T} \|v(t) - v(s)\|_X / (t - s)^\alpha$. If X is merely a topological space, then we still define $C_T X$ but drop the norm.
- $C^1(I, X)$, for some interval $I \subset \mathbb{R}$ and X some Banach space: The space of differentiable functions such that the derivative lies in $C(I, X)$ with continuous extension where the boundary of the interval is closed.
- $C^{1,2}((0, T) \times U)$, where $U \subset \mathbb{R}^2$ is some open set: The space of real-valued functions whose first temporal and second spatial derivatives exist in U and are continuous.
- $S(\mathbb{R}^d)$: The space of tempered distributions.
- $\mathbb{C}$: The complex numbers with imaginary unit $\mathfrak{i}$. $\overline{\cdot}^{\mathbb{C}}$: Complex conjugation.
- $\langle f, g \rangle_{L^2(D,\mathbb{C})} = \int_D f(x) \overline{g(x)}^{\mathbb{C}} dx$, where $D \subset \mathbb{R}^d$, $d \in \mathbb{N}$.
- $\overline{A}$, where A is some subset of a topological space: The closure.
- $C_c(A)$: The continuous functions with compact support in the interior of A.
- $C_0(A)$: The continuous functions vanishing at ∂A.
- $a \lesssim b$: The inequality $a \le Cb$, where $C > 0$ does not depend on a, b. We may include subscripts to emphasize dependencies of C.
- $\mathscr{S}$, $\mathscr{S}_p$: The spectrum and the point spectrum.
- $\mathcal{F}_{\mathbb{T}^d_{2L}}$: The Fourier transform on the torus.
- P_t for $t \ge 0$: The heat semigroup on $[0, L]^2$ with Dirichlet boundary conditions.
- T_t, T^m_t for $t \ge 0$: The semigroup for the continuous (mollified) Parabolic Anderson Model on $[0, L]^2$ with Dirichlet boundary conditions.
- $\mathcal{B}(A)$: The Borel sigma algebra of some topological space A.
- $M(A)$: The space of Radon measures on the Polish space A, equipped with the C_c-vague topology. Integration of a function f with a measure μ is denoted by $\langle f, \mu \rangle$.

- $M_F(A)$: The space of finite Radon measures on the Polish space A, equipped with the C_c-vague topology.
- $M_1(A)$: The space of Borel probability measures on the Polish space A, equipped with the topology of weak convergence.

5.2 Index of Martingales

Let $n, m \in \mathbb{N}$ with n sufficiently large.
• Let $Y^{n,m}$ be a solution to the martingale problem for $(L^y_{n,m}, D(L^y_{n,m}))$, where

$$D(L^y_{n,m}) := \Big\{ \exp(\langle \log(g), \cdot \rangle) | g \in \mathrm{Dom}(G^{\mathfrak{a}}_n), \|g\|_{L^\infty} \leq 1, \inf_{(0,L)^2} g > 0 \Big\}$$

where on this set the generator is given by

$$L^y_{n,m} \exp(\langle \log(g), \cdot \rangle)(\mu) := \left\langle \frac{G^{\mathfrak{a}}_n g + \Phi_{n,m}(g) - g}{g}, \mu \right\rangle \exp(\langle \log(g), \mu \rangle), \quad \mu \in E.$$

Reference: Definition 2.8.
• Let $X^{n,m}_t := 1/n Y^{n,m}_{nt}$ be associated to the martingale problem for $(L_{n,m}, D(L_{n,m}))$.
Reference: Definition 2.12.
• Let X^m be a killed mollified Super Brownian Motion, i.e. a solution to the martingale problem for $(L_m, D(L_m))$, where

$$D(L_m) := \{\exp(- \langle \phi, \cdot \rangle) | \phi \in \mathrm{Dom}(\mathscr{H}_{\boldsymbol{\xi}_m}), \phi \geq 0\}$$

where for $\phi \in \mathrm{Dom}(\mathscr{H}_{\boldsymbol{\xi}_m})$, $\phi \geq 0$, the generator is given by

$$L_m \exp(- \langle \phi, \cdot \rangle)(\mu) := \left\langle -\mathscr{H}_{\boldsymbol{\xi}_m} \phi + \frac{1}{2}\phi^2, \mu \right\rangle \exp(- \langle \phi, \mu \rangle).$$

Reference: Definition 2.34.
• Let X be a killed rough Super Brownian Motion, i.e. a solution to the martingale problem for $(L, D(L))$, where

$$D(L) := \{\exp(- \langle \phi, \cdot \rangle) | \phi \in D_{\mathscr{H}_{\boldsymbol{\xi}}}\}$$

where for $\phi \in D_{\mathscr{H}_{\boldsymbol{\xi}}}$ the generator is given by

$$L \exp(- \langle \phi, \cdot \rangle)(\mu) := \left\langle -\mathscr{H}_{\boldsymbol{\xi}} \phi + \frac{1}{2}\phi^2, \mu \right\rangle \exp(- \langle \phi, \mu \rangle).$$

Reference: Definition 3.1.
For martingales derived from those fundamental processes, we use the following naming convention: L stands for linear martingales, E stands for exponential martingales, indices appear according to the superscript of X, the superscript ϕ represent the function the martingale is associated to, $\tilde{\cdot}$ represents time-dependent martingales, two time variables

(s,t) represents that we used semigroups to remove generators. The superscripts $1,2,3$ refer to the (local) martingales M derived in the proofs of Lemma 2.21 and Lemma 3.3. The martingales below are ordered by first appearance.

• Let $\phi \in C_c^2((0,L)^2)$, $\phi \geq 0$. Then

$$L_{n,m}^{\phi}(t) := \langle \phi, X_t^{n,m} \rangle - \langle \phi, X_0^{n,m} \rangle - \int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m} \phi, X_s^{n,m} \rangle \, ds$$

is a martingale with predictable quadratic variation

$$\langle L_{n,m}^{\phi} \rangle_t = \int_0^t \left\langle \frac{2}{n} \nabla \phi^T \nabla \phi + \phi^2, X_s^{n,m} \right\rangle ds.$$

Reference: Lemma 2.21.

• Further,

$$\begin{aligned} M_{n,m}^{1,\phi}(t) &:= \langle \phi, X_t^{n,m} \rangle^2 - \langle \phi, X_0^{n,m} \rangle^2 \\ &\quad - \int_0^t \left\langle \frac{2}{n} \nabla \phi^T \nabla \phi + \phi^2, X_s^{n,m} \right\rangle + 2 \langle \mathscr{H}_{\boldsymbol{\xi}_m} \phi, X_s^{n,m} \rangle \langle \phi, X_s^{n,m} \rangle \, ds \end{aligned}$$

is a martingale. **Reference:** Proof of Lemma 2.21.

• Further,

$$M_{n,m}^{2,\phi}(t) := \left(\langle \phi, X_t^{n,m} \rangle - \int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m} \phi, X_s^{n,m} \rangle \, ds \right) \langle \phi, X_0^{n,m} \rangle$$

is a martingale. **Reference:** Proof of Lemma 2.21.

• Further,

$$\begin{aligned} M_{n,m}^{3,\phi}(t) &:= 2 \int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m} \phi, X_s^{n,m} \rangle \langle \phi, X_s^{n,m} \rangle \, ds + \left(\int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m} \phi, X_s^{n,m} \rangle \, ds \right)^2 \\ &\quad - 2 \langle \phi, X_t^{n,m} \rangle \int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m} \phi, X_s^{n,m} \rangle \, ds \end{aligned}$$

is a local martingale. **Reference:** Proof of Lemma 2.21.

• Further for $\phi \in \mathrm{Dom}(\mathscr{H}_{\boldsymbol{\xi}_m})$, $\phi \geq 0$, $f_n := 1 - \phi/n$,

$$\begin{aligned} E_{n,m}^{n \log(f_n)}(t) :&= \exp(\langle \log(f_n), n X_t^{n,m} \rangle) - \exp(\langle \log(f_n), n X_0^{n,m} \rangle) \\ &\quad - \int_0^{nt} \left\langle \frac{G_n^{\mathfrak{a}} f_n + \Phi_{n,m}(f_n) - f_n}{f_n}, n X_{s/n}^{n,m} \right\rangle \exp\left(\left\langle \log(f_n), n X_{s/n}^{n,m} \right\rangle \right) ds \end{aligned}$$

is a martingale. **Reference:** Proof of Lemma 2.37.

• Assume that $\phi \in C^1([0,\infty), C_0((0,L)^2))$, $\phi(t) \in \mathrm{Dom}(\mathscr{H}_{\boldsymbol{\xi}_m})$ for any $t > 0$ and $\phi \geq 0$. Also assume that $\mathscr{H}_{\boldsymbol{\xi}_m} \phi \in C([0,\infty), C_0((0,L)^2))$. Then

$$\begin{aligned} \tilde{E}_m^{\phi}(t) &:= \exp(-\langle \phi(t), X_t^m \rangle) - \exp(-\langle \phi(0), X_0^m \rangle) \\ &\quad - \int_0^t \left\langle -\mathscr{H}_{\boldsymbol{\xi}_m} \phi(s) + \frac{1}{2} \phi^2(s) - \partial_s \phi(s), X_s^m \right\rangle \exp(-\langle \phi(s), X_s^m \rangle) ds \end{aligned}$$

is a martingale. **Reference:** Lemma 2.39.
• Let $\psi \in \mathrm{Dom}(\mathscr{H}_{\boldsymbol{\xi}_m}) \cap \mathcal{C}_{\mathfrak{0}}^{(\alpha+2)+}$, $\psi \geq 0$, $\psi \neq 0$, $t > 0$, and $\gamma > 0$ be sufficiently small. Then it holds that the process for $s \leq t$ given by

$$E_m^{\gamma\psi}(s,t) := \exp(-\langle U_{t-s}^m(\gamma\psi), X_s^m\rangle) - \exp(-\langle U_t^m(\gamma\psi), X_0^m\rangle)$$

is a bounded martingale. **Reference:** Lemma 2.40.
• Assume that $\phi \in C^1([0,\infty), C_0((0,L)^2))$, $\phi(t) \in \mathrm{Dom}(\mathscr{H}_{\boldsymbol{\xi}_m})$ for any $t > 0$ and $\phi \geq 0$. Also assume that $\mathscr{H}_{\boldsymbol{\xi}_m}\phi \in C([0,\infty), C_0((0,L)^2)$. Then

$$\tilde{L}_m^{\phi}(t) = \langle \phi(t), X_t^m\rangle - \langle \phi(0), X_0^m\rangle - \int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi(s) + \partial_s\phi(s), X_s^m\rangle\, ds$$

is a continuous, square-integrable martingale with quadratic variation

$$\langle \tilde{L}_m^{\phi}\rangle_t = \int_0^t \langle \phi^2(s), X_s^m\rangle ds.$$

Reference: Lemma 3.3.
• Further,

$$\tilde{M}_m^{1,\phi}(t) := \langle \phi(t), X_t^m\rangle^2 - \langle \phi(0), X_0^m\rangle^2$$
$$- \int_0^t \langle \phi^2(s), X_s^m\rangle + 2\langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi(s) + \partial_s\phi(s), X_s^m\rangle \langle \phi(s), X_s^m\rangle\, ds$$

is a martingale. **Reference:** Lemma 3.3.
• Further,

$$\tilde{M}_m^{2,\phi}(t) := \left(\langle \phi(t), X_t^m\rangle - \int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi(s) + \partial_s\phi(s), X_s^m\rangle\, ds\right)\langle \phi(0), X_0^m\rangle$$

is a martingale. **Reference:** Lemma 3.3.
• Further,

$$\tilde{M}_m^{3,\phi}(t) := 2\int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi(s) + \partial_s\phi(s), X_s^m\rangle \langle \phi(s), X_s^m\rangle\, ds$$
$$+ \left(\int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi(s) + \partial_s\phi(s), X_s^m\rangle\, ds\right)^2 - 2\langle \phi(t), X_t^m\rangle \int_0^t \langle \mathscr{H}_{\boldsymbol{\xi}_m}\phi(s) + \partial_s\phi(s), X_s^m\rangle\, ds$$

is a local martingale. **Reference:** Lemma 3.3.
• Let $t > 0$, $\phi \in \mathrm{Dom}(\mathscr{H}_{\boldsymbol{\xi}_m})$, $\phi \geq 0$. The process for $s \leq t$ given by

$$L_m^{\phi}(s,t) := \langle T_{t-s}^m\phi, X_s^m\rangle - \langle T_t^m\phi, X_0^m\rangle$$

is a continuous, square-integrable martingale with quadratic variation given by

$$\langle L_m^{\phi}(\cdot,t)\rangle_s = \int_0^s \langle (T_{t-r}^m\phi)^2, X_r^m\rangle dr.$$

Reference: Lemma 3.4.
• Further,

$$M^{1,\phi}_m(s,t) := \langle T^m_{t-s}\phi, X^m_s\rangle^2 - \langle T^m_t\phi, X^m_0\rangle^2 - \int_0^s \langle (T^m_{t-r}\phi)^2, X^m_r\rangle dr$$

is a martingale. **Reference:** Lemma 3.4.
• Let $\phi \in C^1([0,\infty), D_{\mathscr{H}_\xi})$ be such that $\mathscr{H}_\xi\phi \in C([0,\infty), \mathcal{C}^{\alpha+2}_{\mathfrak{d}})$. Then,

$$\tilde{L}^\phi(t) := \langle \phi(t), X_t\rangle - \langle \phi(0), X_0\rangle - \int_0^t \langle \mathscr{H}_\xi\phi(s) + \partial_s\phi(s), X_s\rangle\, ds$$

is a continuous, square-integrable martingale with quadratic variation

$$\langle \tilde{L}^\phi\rangle_t = \int_0^t \langle \phi(s)^2, X_s\rangle ds.$$

Reference: Lemma 3.10.
• Let $t > 0$, $u_0 \in \mathcal{C}^{(\alpha+2)+}_{\mathfrak{d}}$, $u_0 \geq 0$, $u_0 \neq 0$, and $f \in C([0,t], \mathcal{C}^{(\alpha+2)+}_{\mathfrak{d}})$, $f \geq 0$, $f \neq 0$. Let u^t be a mild solution to

$$\begin{cases} (\partial_s + \mathscr{H}_\xi)u^t = f \text{ in } (0,t) \times (0,L)^2, \\ u^t(t) = u_0 \text{ in } [0,L]^2,\ u^t = 0 \text{ on } [0,t] \times \partial[0,L]^2, \end{cases}$$

as constructed in Section 1.8. Then it holds that for $s \leq t$,

$$L^{u_0,f}(s,t) := \langle u^t(s), X_s\rangle - \langle u^t(0), X_0\rangle - \int_0^s \langle f(r), X_r\rangle dr$$

is a continuous, square-integrable martingale with quadratic variation

$$\langle L^{u_0,f}(\cdot,t)\rangle_s = \int_0^s \langle (u^t(r))^2, X_r\rangle dr.$$

Reference: Lemma 3.11.
• Let $\psi \in \mathcal{C}^{(\alpha+2)+}_{\mathfrak{d}}$, $\psi \geq 0$, $\psi \neq 0$, $t > 0$, and $\gamma > 0$ be small enough. Then it holds that the process for $s \leq t$ given by

$$E^{\gamma\psi}(s,t) := \exp(-\langle U_{t-s}(\gamma\psi), X_s\rangle) - \exp(-\langle U_t(\gamma\psi), X_0\rangle)$$

is a continuous, bounded martingale. **Reference:** Lemma 3.12.

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